Institute of Computer Science of CAS

Modeling rivulets and other thin film flows

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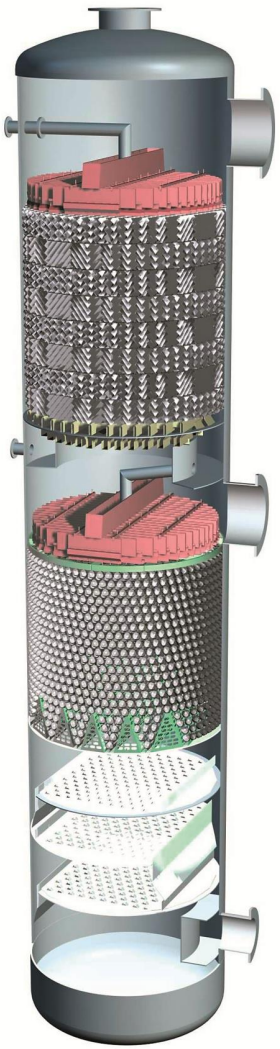
17. 2. 2015

1 Introduction

- Why to study rivulet interface
- General problem description

Why to study rivulets

Numerous applications in mass transfer and reaction engineering



[Sulzer ChemTech]

Hydrodynamics

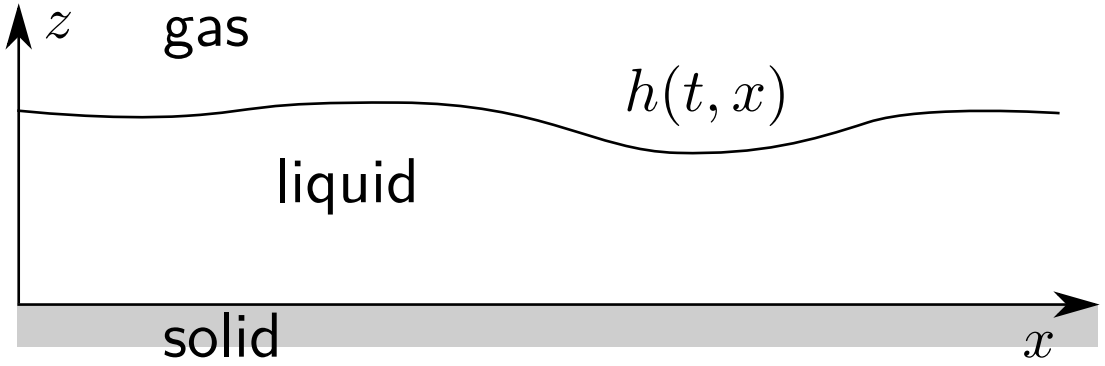
- Fuel cells
  - water management inside PEMFC fuel cells
- Aerospace engineering
  - in flight formation of rivulets on plane wings

Gas-liquid interface

- Packed columns
  - wetting performance
  - mass transfer coefficients
- Catalytic reactors
  - wetting of the catalyst

General problem description

Thin film flow on solid substrate[1]



Scales

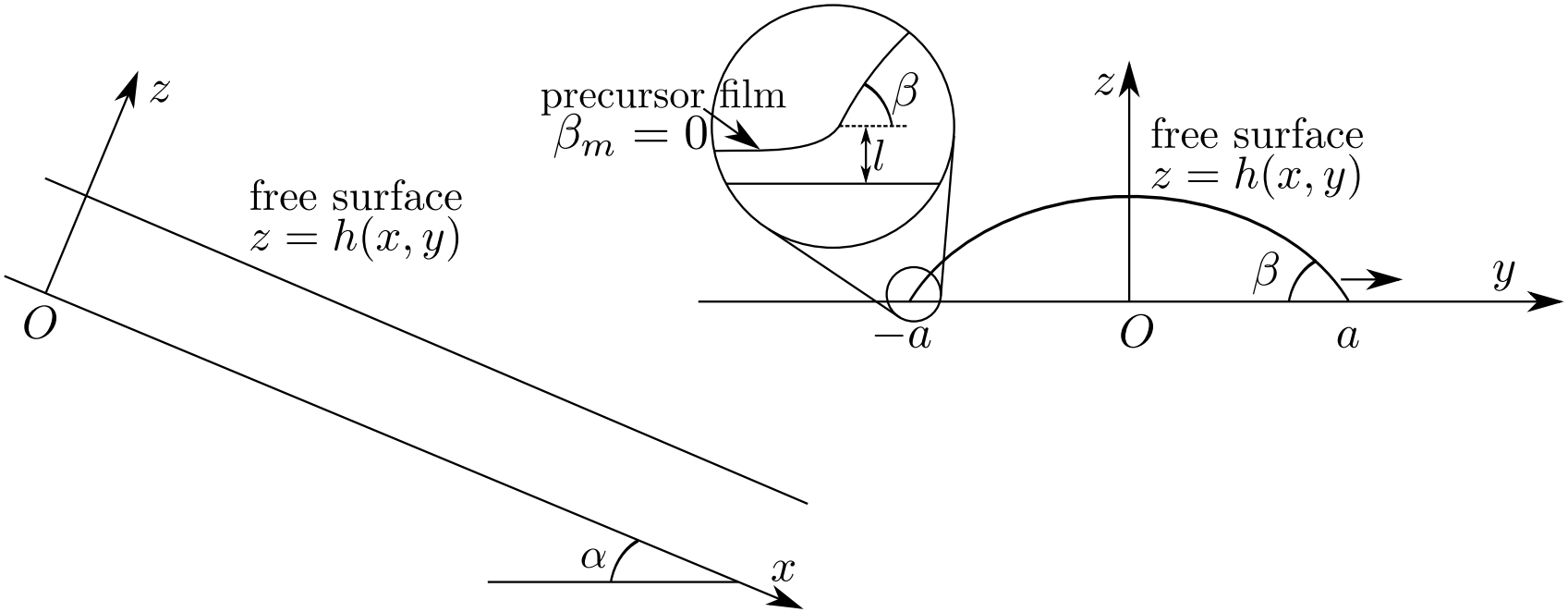
- Thickness, H
- General lengthscale, L
- $\varepsilon = H/L \ll 1$

Flow

- Predominantly in direction of one of the longer dimensions
- Driven by external forces (gravity, surface tension gradients...)
- Modeled via macroscopic momentum equation (Navier-Stokes)

Used coordinate system

Cartesian coordinate system and basic notations



Notations

a .... half-width of the rivulet, $[\mathrm{m}]$
h ..... height, $[\mathrm{m}]$
l ..... intermediate region length scale, $[\mathrm{m}]$

x, y, z .... coordinate system, $[\mathrm{m}]$
α ..... plate inclination angle, $[^\circ]$
β ..... dynamic contact angle, $[^\circ]$

Navier-Stokes equations

Macroscopic momentum balance for incompressible liquid

$$\rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}\,t} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{F}$$

Legend and notations

$(t, \mathbf{x}) = (t, x, y, z)$ .... coordinates
$\mathbf{u} = (u, v, w)$ ..... liquid velocity
p ..... pressure
ρ ..... liquid density
μ ..... liquid dynamic viscosity
$\mathbf{F}$ ..... external forces

$$\frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}\,t} = \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}$$

2 Thin Film I

- Main Idea
- Reduced Navier-Stokes equations
- Nondimensionalization
- Leading terms analysis
- Boundary conditions

Main idea of thin film approximation

Thin liquid layer moving in one direction subject to external forces

Simplifications

- Newtonian liquid, ρ, μ and γ are constant
- Gravity is the only acting body force.
- No shear at gas-liquid interface.
- Laminar flow.

General approach

- Perform dimension analysis.
- Neglect as many of terms in Navier-Stokes equation as possible.
- Integrate NS equation partially to obtain a scalar PDE.

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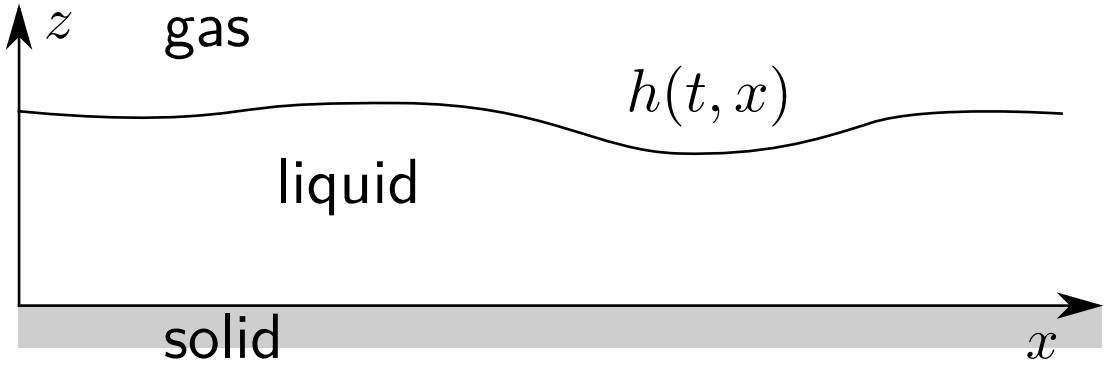
Reduction of Navier-Stokes equations

Case of 1D thin film flow on horizontal substrate



$$\rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}\,t} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{F}$$

$$\rho(u_t + u u_x + w u_z) = -p_x + \mu\left(u_{xx} + u_{zz}\right)$$
$$\rho(w_t + u w_x + w w_z) = -p_z + \mu\left(w_{xx} + w_{zz}\right) + g$$

Dimensionless formulation

Introduction of scales for the problem

$$\bar{x} = \frac{x}{L}, \quad \bar{z} = \frac{z}{\varepsilon L}, \quad \bar{u} = \frac{u}{U}, \quad \bar{w} = \frac{w}{\varepsilon U}, \quad \bar{p} = \frac{\varepsilon^2 L}{\mu U^2} p, \quad \varepsilon = \frac{H}{L}$$

$$\rightsquigarrow$$

$$\begin{aligned}
\frac{U^2}{L}\left(\bar{u}_t + \bar{u}\bar{u}_x + \bar{w}\bar{u}_z\right) &= \frac{\mu U}{\varepsilon^2 \rho L^2}\left(-\bar{p}_x + \varepsilon^2 \bar{u}_{xx} + \bar{u}_{zz}\right) \\
\varepsilon \frac{U^2}{L}\left(\bar{w}_t + \bar{u}\bar{w}_x + \bar{w}\bar{w}_z\right) &= \frac{\mu U}{\varepsilon^3 \rho L^2}\left(-\bar{p}_z + \varepsilon^4 \bar{w}_{xx} + \varepsilon^2 \bar{w}_{zz}\right) + g
\end{aligned}$$

Reynolds number – ratio of inertial to viscous forces

$$\mathrm{Re} = \frac{\rho L U}{\mu}$$

Leading term analysis

Neglect less significant terms

$$
\begin{aligned}
\varepsilon^2 \mathrm{Re}\left(\bar{u}_t + \bar{u}\bar{u}_x + \bar{w}\bar{u}_z\right) &= -\bar{p}_x + \varepsilon^2 \bar{u}_{xx} + \bar{u}_{zz} \\
\varepsilon^4 \mathrm{Re}\left(\bar{w}_t + \bar{u}\bar{w}_x + \bar{w}\bar{w}_z\right) &= -\bar{p}_z + \varepsilon^4 \bar{w}_{xx} + \varepsilon^2 \bar{w}_{zz} + g,
\end{aligned}
$$

Thin film and laminar flow

$$\varepsilon \ll 1, \qquad \varepsilon^2 \mathrm{Re} = \varepsilon^2 \frac{\rho L U}{\mu} \ll 1$$

Simplified NS equations in dimensional form

$$
\begin{aligned}
p_x &= \mu u_{zz} \\
p_z &= \rho g
\end{aligned}
$$

Boundary conditions

No slip, no shear stress, pressure jump at the $(g) - (l)$ interface and kinetic condition

Dirichlet and Neumann's conditions

- No slip at the $(l) - (s)$ interface, $u(0) = w(0) = 0$.
- No shear at the free surface, $u_z(h) = 0$.

Pressure jump at free boundary

$$p(h) = p_A - \gamma\kappa$$

Kinematic boundary condition

$$F(t,x) = h(t,x) - z = 0$$

$$\frac{\mathrm{D}\,F}{\mathrm{D}\,t} = 0 \iff h_t + uh_x = w$$

Continuity equation – mass balance

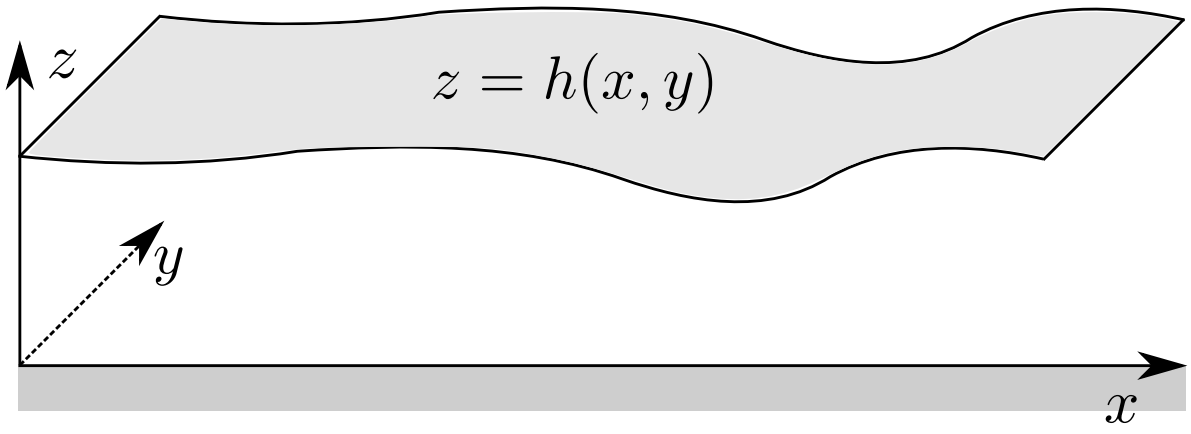
$$u_x + w_z = 0 \iff w(h) - w(0) = w(h) = \int_0^h u_x \mathrm{d}\,z$$

3 Surfaces

- Monge parametrization
- Mean surface curvature

Monge parametrization

Natural surface parametrization for thin liquid films[2]



Surface definition

$$\mathbf{r} = (x, y, h(x, y)) \iff F(x, y, z) = z - h(x, y) = 0$$

Tangent plane and outer unit normal (smooth surfaces)

$$\mathbf{r} \cdot \mathbf{n} = 0, \quad \mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\|\mathbf{r}_x \times \mathbf{r}_y\|} = \left.\frac{\nabla F}{\|\nabla F\|}\right|_{z=h(x,y)} \tag{1}$$

Curvature tensor

Expressing the surface curvature

Curvature

Linear object: Change in tangent along the arc length of the curve.

Curve on surface: Change in $\mathbf{n}$ with movement on the surface.

$$\mathrm{d}\mathbf{n} = \mathrm{d}\mathbf{r} \cdot \mathbf{Q}$$

Curvature tensor (elements from differentiation of (1))

$$Q_{ij} = \frac{1}{\Gamma}\left(F_{ij} - \frac{F_i \Gamma_j}{\Gamma}\right)$$

$$\Gamma = \|\nabla F\|, \quad F_i = \frac{\partial F}{\partial r_i}, \quad \Gamma_i = \frac{\partial \|\nabla F\|}{\partial r_i}, \quad \mathbf{r} = (x, y, z)$$

Curvature tensor invariants

Invariants of Q do not change with rotation of coordinate system[2, 3]

Matrix 3×3, Q, invariants under similarity transformations

- Mean curvature, $\kappa = \mathrm{Tr}\, Q/2$
- Gaussian curvature, $K = M_{11} + M_{22} + M_{33}$
- $\det Q = 0$

Mean curvature

$$\kappa = \frac{1}{2\Gamma^3}\left[F_{xx}(F_y^2 + F_z^2) - 2F_xF_yF_{xy} + \mathrm{Perm}\right] \tag{2}$$

$$\mathrm{Perm} : (x, y, z) \to (z, x, y), \quad (x, y, z) \to (y, z, x)$$

Simplification of (2) via Monge parametrization

$$F = z - h(x, y), \quad \Gamma = \|\nabla F\| = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$\kappa = \frac{(1 + h_x^2)h_{yy} + (1 + h_y^2)h_{xx} - 2h_xh_yh_{xy}}{2(1 + h_x^2 + h_y^2)^{\frac{3}{2}}}$$

4 Thin Film II

- Remainder
- Partial integration
- Comments on the result

Where we were left

Simplified Navier-Stokes equations and boundary conditions

Navier-Stokes equations

$$p_x = \mu u_{zz} \tag{3}$$

$$p_z = \rho g \tag{4}$$

Boundary conditions

$$u(0) = w(0) = 0 \tag{5}$$

$$u_z(h) = 0 \tag{6}$$

$$p(h) = p_A - \gamma\kappa \tag{7}$$

$$h_t + uh_x = w \tag{8}$$

$$w(h) = \int_0^h u_x \mathrm{d}\, z \tag{9}$$

Goal

Partially integrate equations (3) and (4) to obtain single PDE.

Partial integration I

Obtaining the velocity profile

Integration of (4) and usage of BC (7)

$$\int_z^h p_z \mathrm{d}z = \int_z^h \rho g \mathrm{d}z \rightsquigarrow p(h) - p(z) = \rho g (h - z)$$

$$p(z) = p_A + \rho g (z - h) - \gamma \kappa \tag{10}$$

Substitution from (10) into (3) and integration

$$p_x = -\rho g h_x + \gamma \kappa_x = \mu u_{zz}$$

$$\int_z^h p_x \mathrm{d}z = \mu \int_z^h u_{zz} \mathrm{d}z$$

$$p_x h - p_x z = \mu \left(u_z(h) - u_z(x) \right) \tag{11}$$

Introduction of no-shear condition (6) and integration of (11)

$$p_x h z - p_x \frac{z^2}{2} = -\mu \int_0^z u_z \mathrm{d}z = -\mu \left(u(z) - u(0) \right) = -\mu u(z) \tag{12}$$

Partial integration II

Deriving single PDE for $h(t,x)$

Combination of kinematic BC (8) and mass balance (9)

$$h_t + uh_x = -\int_0^h u_x \mathrm{d}z \tag{13}$$

Velocity field and its derivation with respect to x

$$u(z) = \frac{1}{\mu}\left(p_x \frac{z^2}{2} - p_x h z\right)$$

$$u_x(z) = \frac{1}{\mu}\left(p_{xx} \frac{z^2}{2} - p_{xx} h z - p_x h_x z\right)$$

Substitution for $u(z)$ and $u_x(z)$ to (13) and integration

$$h_t + \frac{1}{\mu}\left(p_x \frac{z^2}{2} - p_x h z\right) h_x = -\int_0^h \frac{1}{\mu}\left(p_{xx} \frac{z^2}{2} - p_{xx} h z - p_x h_x z\right) \mathrm{d}z$$

$$h_t - \frac{1}{3\mu}\frac{\partial}{\partial x}\left(p_x h^3\right) = h_t - \frac{1}{3\mu}\frac{\partial}{\partial x}\left[h^3\left(\rho g h_x - \gamma \kappa_x\right)\right] = 0 \tag{14}$$

Comments on the result

Further possible simplifications

Derived equation

$$h_t - \frac{1}{3\mu}\frac{\partial}{\partial x}\left[h^3\left(\rho g h_x - \gamma \kappa_x\right)\right] = 0$$

Case of nearly flat surface with neglectable effects of gravity

- Nearly flat surface, $h_x \ll 1 \rightsquigarrow \kappa_x \sim h_{xxx}$
- Neglectable gravity, $\rho g \sim 0$

Simplified equation

$$h_t + \frac{\gamma}{3\mu}\frac{\partial}{\partial x}\left(h^3 h_{xxx}\right) = 0$$

5 Spreading

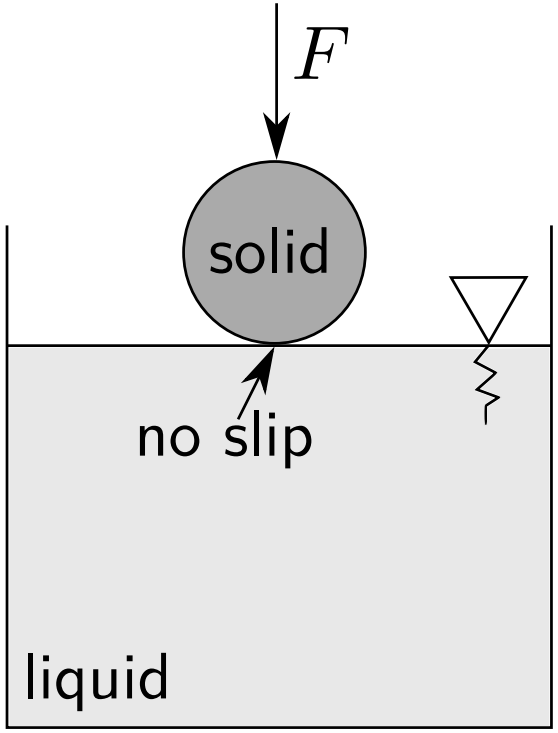
- Hamlet
- Coping with Huh and Scriven paradox
- Cox-Voinov law
- To do

Something is rotten in the state of Denmark

Liquid spreading and no slip boundary condition

Derived equation – assumption of no slip at $z = 0$

$$h_t - \frac{1}{3\mu}\frac{\partial}{\partial x}\left[h^3\left(\rho g h_x - \gamma \kappa_x\right)\right] = 0 \iff u(0) = w(0) = 0$$



Huh and Scriven paradox[4]

Even Heracles could not sink a solid.

$$\mathbf{u}(0) = 0 \iff F \to \infty$$

Coping with Huh and Scriven paradox

Slip or no slip – that is the question[6]

Navier slip

$$z = 0, \qquad \mathbf{u} - \lambda \frac{\partial}{\partial z} \mathbf{u} = 0$$

$$\rightsquigarrow$$

$$h_t - \frac{1}{3\mu} \frac{\partial}{\partial x} \left[\left(h^3 + 3\lambda h^2 \right) \left(\rho g h_x - \gamma \kappa_x \right) \right] = 0$$

Disjoining pressure

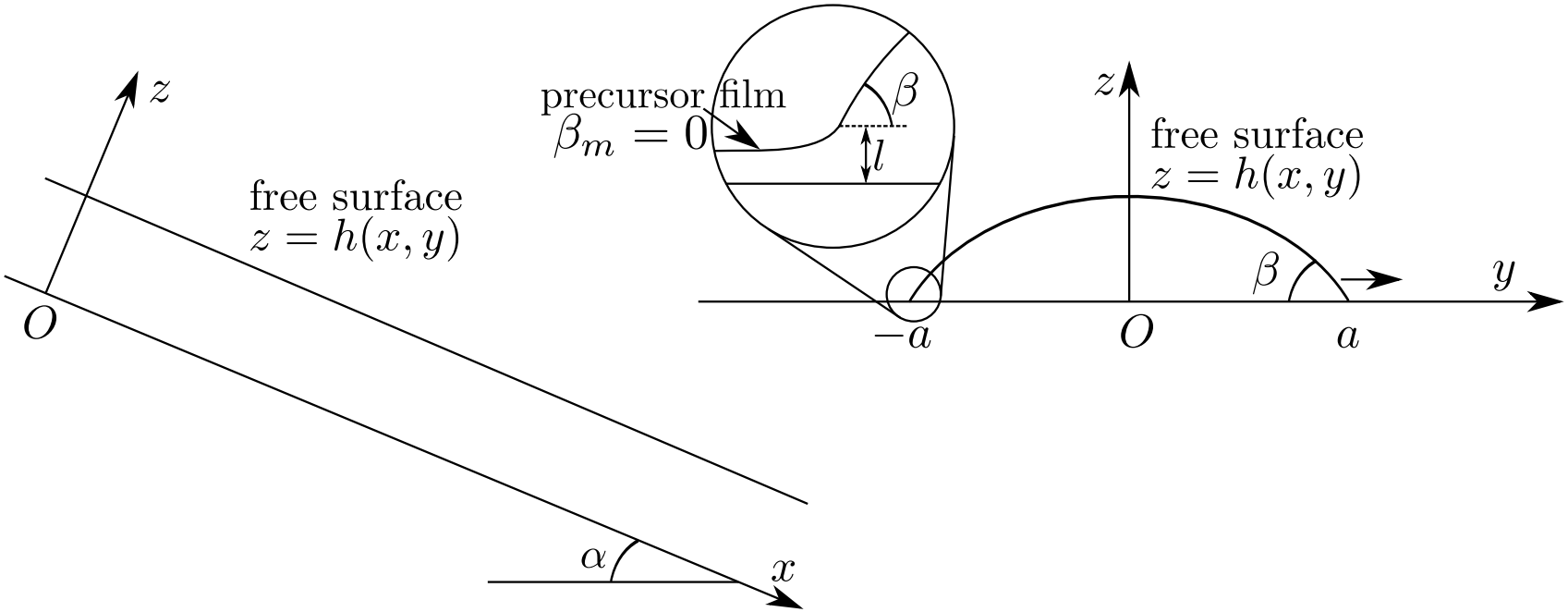
Introduction of new pressure term reflecting the long range molecular forces

$$\rightsquigarrow$$

$$h_t - \frac{1}{3\mu} \frac{\partial}{\partial x} \left[h^3 \left(\rho g h_x - \gamma \kappa_x - \phi_x \right) \right] = 0$$

Used coordinate system

Cartesian coordinate system and basic notations



Notations

a ..... half-width of the rivulet, $[\mathrm{m}]$
h ..... height, $[\mathrm{m}]$
l ... intermediate region length scale, $[\mathrm{m}]$

x, y, z ..... coordinate system, $[\mathrm{m}]$
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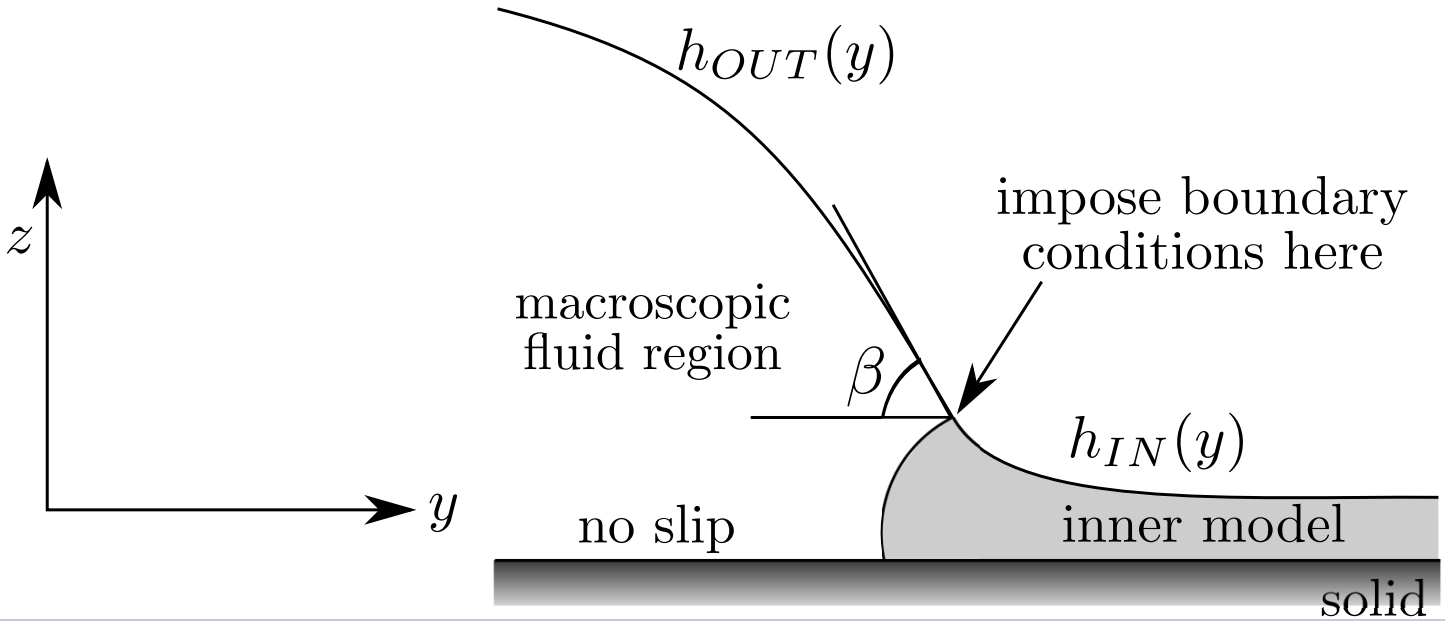
Cox-Voinov law for perfectly wetting liquid

Solution of thin film governing equation for spreading of symmetric object[6]

Thin film governing equation - outer and inner

$$h_t + \frac{\gamma}{3\mu}\frac{\partial}{\partial a}(h^3 h_{aaa}) = 0, \qquad h_t + \frac{1}{3\mu}\frac{\partial}{\partial a}\left[\left(h^3 + 3\lambda h^2\right)\gamma h_{aaa}\right] = 0$$



Cox-Voinov law[6]

$$\beta(t)^3 = 9\frac{\mathrm{d}a(t)}{\mathrm{d}t}\frac{\mu}{\gamma}\ln\left(\frac{a(t)}{2e^2 l}\right)$$

What do we want to achieve

Description of spreading rivulet flow

Uniform (non spreading) rivulet flow

- Same principle of description as for thin film.
- Two-dimensional problem.
- Analytical solution of Navier-Stokes equations is available for simplified cases.

Spreading rivulet

- Three-dimensional problem.
- Problem of gas-liquid interface shape evolution.
- There is no analytical solution available.

Plan

Use available description of uniform rivulet to deal with spreading one.

6 Uniform rivulet

- NS equations
- Notes on integration
- Case (ii)
- Case (i)
- Case (iii)
- Comparison of profiles

Simplified Navier-Stokes equations

Assumption of very shallow and nearly flat rivulet

Simplified NS

$$0 = -p_x + \rho g \sin\alpha + \mu u_{zz} \quad (15)$$

$$0 = -p_y \quad (16)$$

$$0 = -p_z - \rho g \cos\alpha \quad (17)$$

Used boundary conditions

$$\begin{array}{rcl} z = 0: & u = u(y,z) = 0 & \\ z = h: & p = p_A - \gamma h'' & \text{and} \quad u_z = 0 \\ y = \pm a: & h = 0 & \text{and} \quad h' = \pm \tan\beta \end{array}$$

Notes

$$h = h(y), \quad h' = \frac{\mathrm{d}\,h}{\mathrm{d}\,y}, \quad p = p(y,z)$$

Integration of Navier-Stokes equations

In the presented case, there exist a solution for $\alpha \in (-\pi, \pi)$

Express everything as a function of $h(y)$

- Use no-slip, pressure jump and no shear stress boundary conditions.
- Reduce the problem to one, third order, ODE for uknown function $h(y)$.
- Solve this ODE for boundary conditions specifying rivulet edges.

Notation

As it will be shown, the problem will decompose to three cases which will be denoted:

- $(i) \iff \alpha \in (0, \pi/2)$
- $(ii) \iff \alpha = \pi/2$
- $(iii) \iff \alpha \in (\pi/2, \pi)$

Case (ii) I

Vertical plate, no effect of gravity on gas-liquid interface shape

Navier-Stokes equations

$$0 = \rho g + \mu u_{zz} \tag{18}$$

$$0 = -p_y \tag{19}$$

$$0 = -p_z \tag{20}$$

Velocity field – from (18)

$$\int_z^h u_{zz}\,\mathrm{d}z = -\frac{\rho g}{\mu}\int_z^h \mathrm{d}z \quad \Bigg| \; z = h:\; u_z = 0$$

$$\int_0^z u_z\,\mathrm{d}z = \frac{\rho g}{\mu}\int_0^z (h-z)\,\mathrm{d}z \quad \Bigg| \; z = 0:\; u = 0$$

$$u(y,z) = \frac{\rho g}{2\mu}(2hz - z^2)$$

Case (ii) II

Vertical plate, no effect of gravity on gas-liquid interface shape

Pressure field – from (20)

$$\int_z^h p_z \,\mathrm{d}z = \int_z^h 0 \,\mathrm{d}z \quad \Big| \; z = h : \; p = p_A - \gamma h''$$

$$p(y,z) = p_A - \gamma h''$$

$$(19) \rightsquigarrow p_y = -\gamma h''' = 0$$

Gas-liquid interface shape – from (19)

$$\gamma h''' = 0 \rightsquigarrow h(y) = C_1 y^2 + C_2 y + C_3$$

$$y = \pm a : \; h = 0, \quad h' = \pm \tan\beta$$

$$h(\zeta) = \frac{a}{2}(1-\zeta^2)\tan\beta \quad \Big| \quad \zeta = \frac{y}{a}$$

Case (ii) III

Vertical plate, no effect of gravity on gas-liquid interface shape

Volumetric flow rate, Q

$$Q = \int_{-a}^{a} \int_{0}^{h(y)} u(y,z)\,\mathrm{d}z\,\mathrm{d}y$$

Results overview

$$p(z) = p_A + \frac{\gamma}{a}\tan\beta$$

$$h(y) = \frac{\tan\beta}{2a}(a^2 - y^2)$$

$$a(Q) = \left(\frac{105\mu Q}{4\rho g \tan\beta}\right)^{1/4}$$

Case (i) I

Inclined plate, rivulet on top of it

Gas-liquid interface shape – from (16)

$$\gamma h''' - \frac{\rho g \cos\alpha}{\gamma} h' = 0 \quad \Big| \; B = a\sqrt{\frac{\rho g |\cos\alpha|}{\gamma}}$$

$$h(\zeta) = C_1 + C_2 \mathrm{e}^{B\zeta} + C_3 \mathrm{e}^{-B\zeta} \quad \Big| \; \zeta = \frac{y}{a}$$

Application of boundary conditions

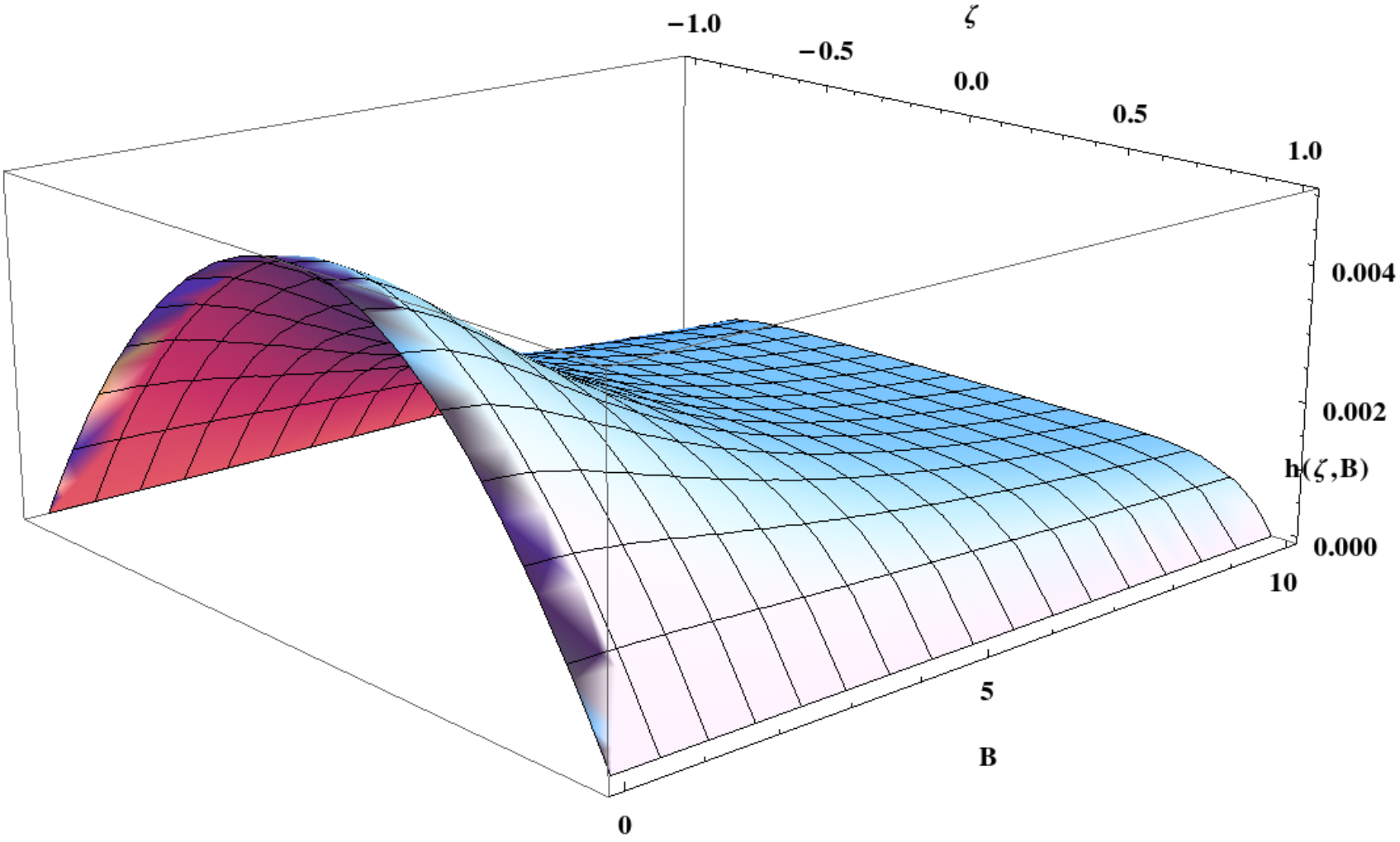
$$y = \pm a : \; h = 0$$

$$y = \pm a : \; h' = \mp \tan\beta$$

$$h(\zeta) = \frac{a \tan\beta}{B} \left(\frac{\cosh B - \cosh B\zeta}{\sinh B} \right)$$

Case (i) II

Gas-liquid interface shape in dependence on Bond number, $a = \beta = 0.1$



Case (iii) I

Inclined plate, rivulet underneath it

Gas-liquid interface shape – from (16)

$$\gamma h''' + \frac{\rho g \cos\alpha}{\gamma} h' = 0 \quad \Big| \, B = a\sqrt{\frac{\rho g |\cos\alpha|}{\gamma}}$$

$$h(\zeta) = C_1 + C_2 \sin B\zeta + C_3 \cos -B\zeta \quad | \, \zeta = \frac{y}{a}$$

Application of boundary conditions

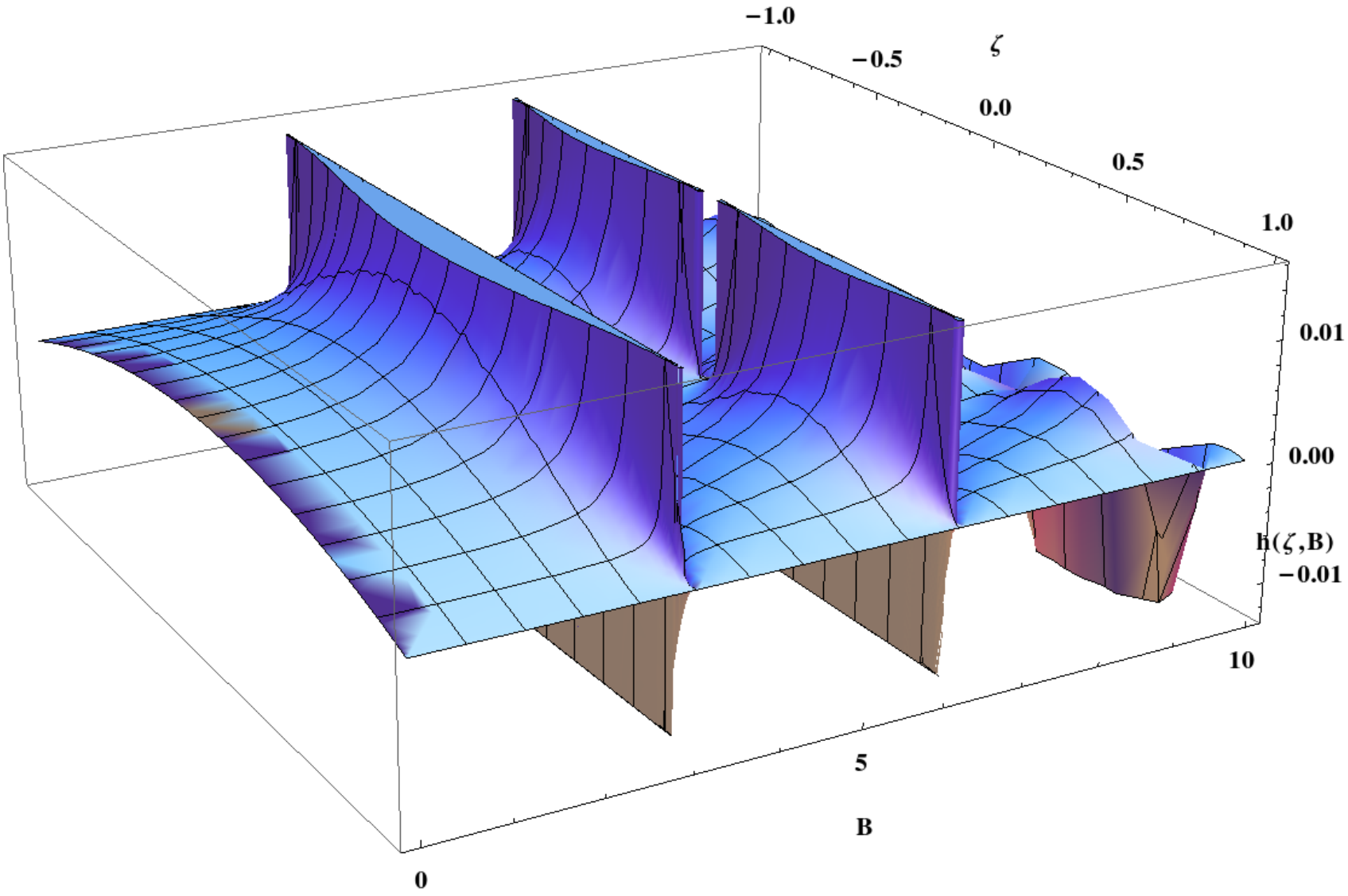
$$y = \pm a : \; h = 0$$

$$y = \pm a : \; h' = \mp \tan\beta$$

$$h(\zeta) = \frac{a \tan\beta}{B} \left(\frac{\cos B\zeta - \cos B}{\sin B} \right)$$

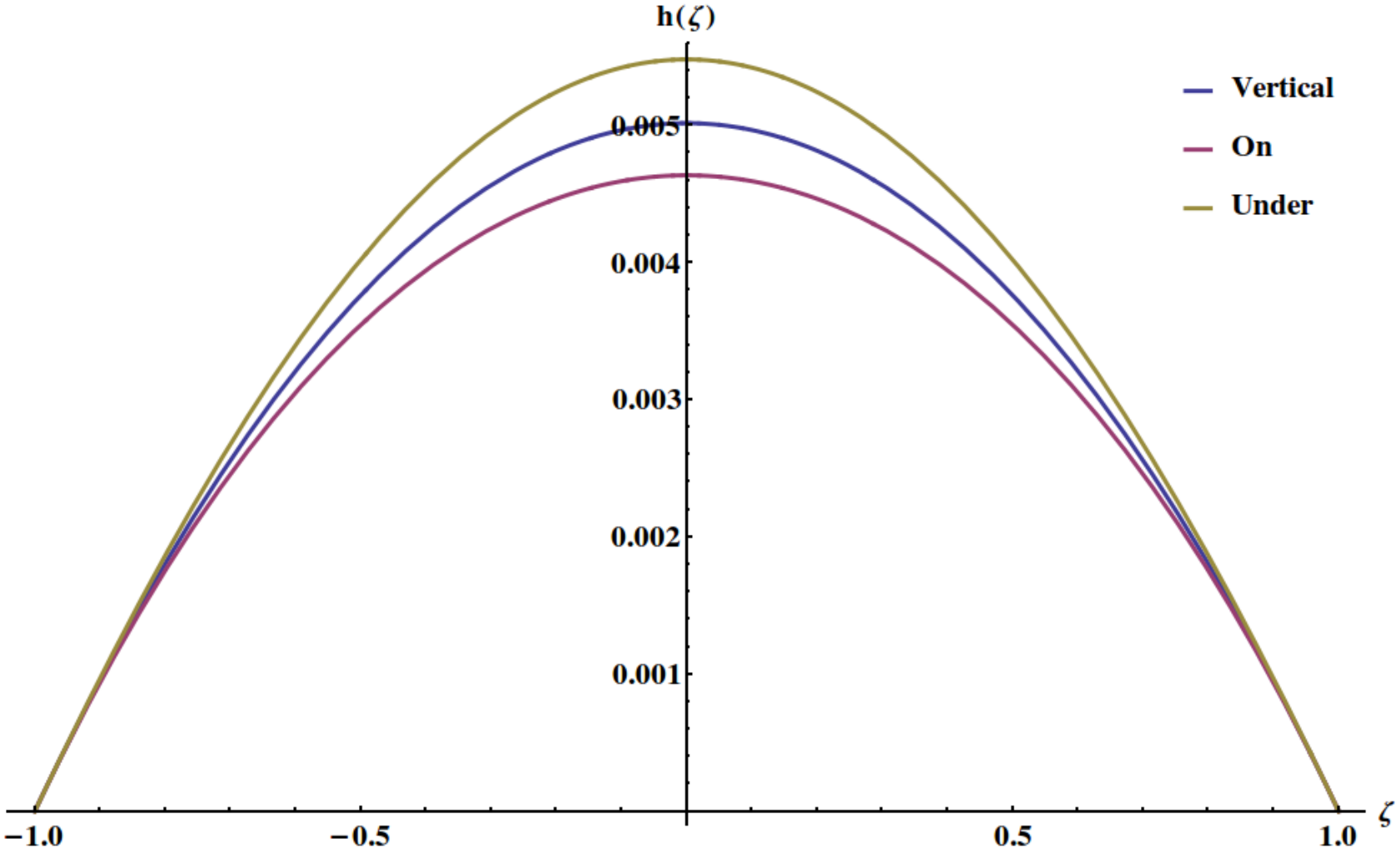
Case (iii) II

Gas-liquid interface shape in dependence on Bond number, $a = \beta = 0.1$



Comparison of uniform rivulet profiles

$B = 1$, $a = \beta = 0.1$, plate inclination angle is included in B



7 Spreading rivulet

- Assumptions
- Basic principle
- Gas-liquid interface shape
- Calculation algorithm

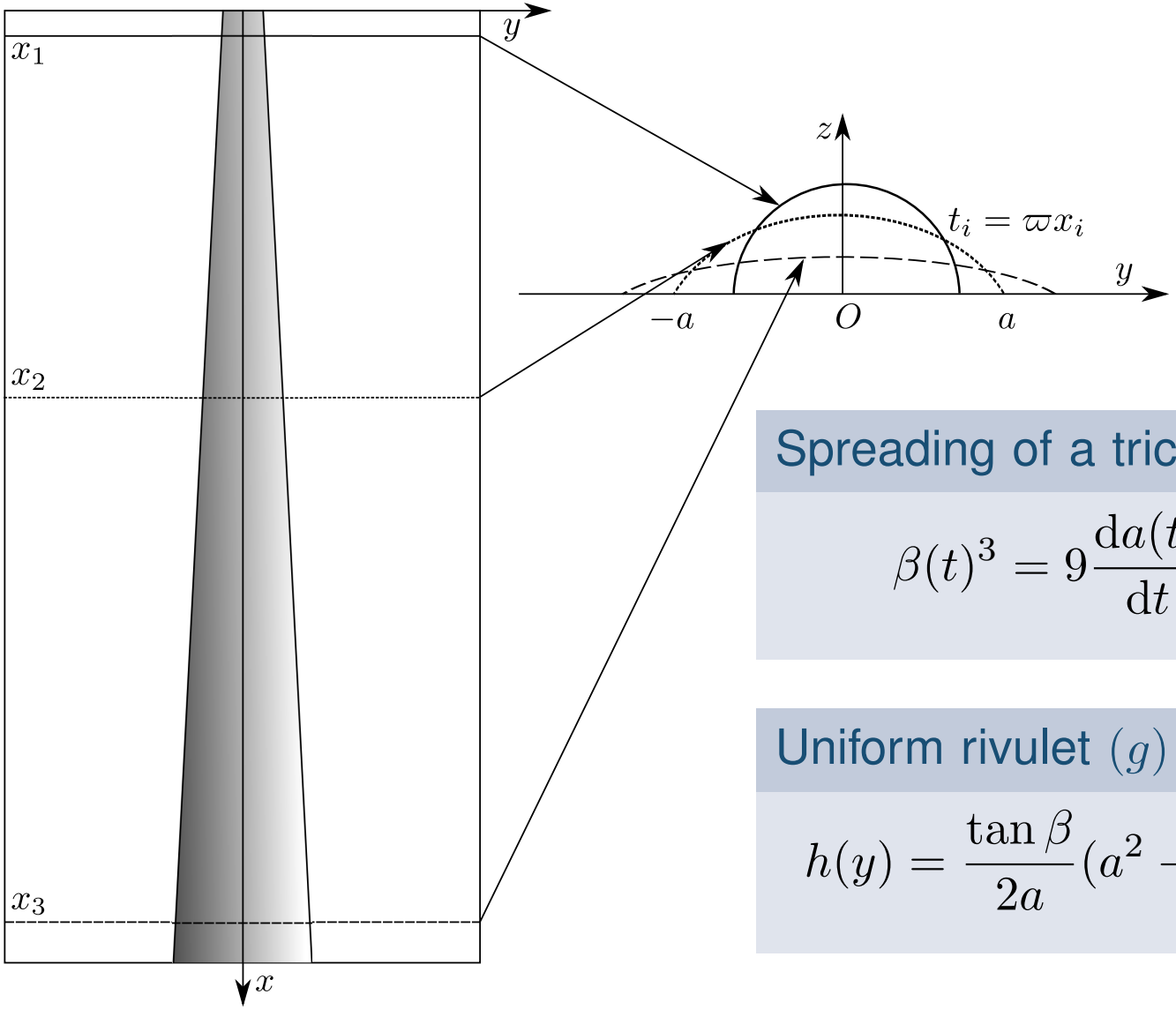
Symplifying assumptions

Reduce problem to one spatial and one time coordinate

- Newtonian liquid, ρ, μ and γ are constant
- $h_t(t, x, y) = 0$, Q is constant
- $\mathbf{u} = (u, v, w)$, $u \gg v \sim w$
- $z = h : u_x = v_y = 0$
- Gravity is the only acting body force.
- Gravity effects on $(g) - (l)$ interface shape can be neglected and $\beta = \beta(x) \ll 1$.
- There is a thin precursor film of height l on the whole studied surface. Thus there is no contact angle hysteresis and $\beta_m = 0$. The height of the precursor film, l, can also be taken as the intermediate region length scale well separating the inner and outer solution for the profile shape[6].

Basic principle

Combination of Cox-Voinov law[6] with research of Duffy and Moffat[7]



Spreading of a trickle in time

$$\beta(t)^3 = 9\frac{\mathrm{d}a(t)}{\mathrm{d}t}\frac{\mu}{\gamma}\ln\left(\frac{a(t)}{2e^2 l}\right)$$

Uniform rivulet $(g)-(l)$ interface

$$h(y) = \frac{\tan\beta}{2a}(a^2 - y^2), \quad a \approx \eta\frac{1}{\beta^{3/4}}$$

Spreading of flowing rivulet in time

Substituting for $a(t) = \eta/\beta(t)^{3/4}$ into Cox-Voinov law and solving arising ODE

First order ODE with separable variables

$$\beta^{19/4} = -A\frac{\mathrm{d}\beta}{\mathrm{d}t}\ln\left(\frac{B}{\beta^{3/4}}\right), \quad \beta = \beta(t), \quad A = \frac{27}{4}\frac{\eta\mu}{\gamma}, \quad B = \frac{\eta}{2e^2 l}$$

Implicit dependence of rivulet $(g) - (l)$ interface shape on time

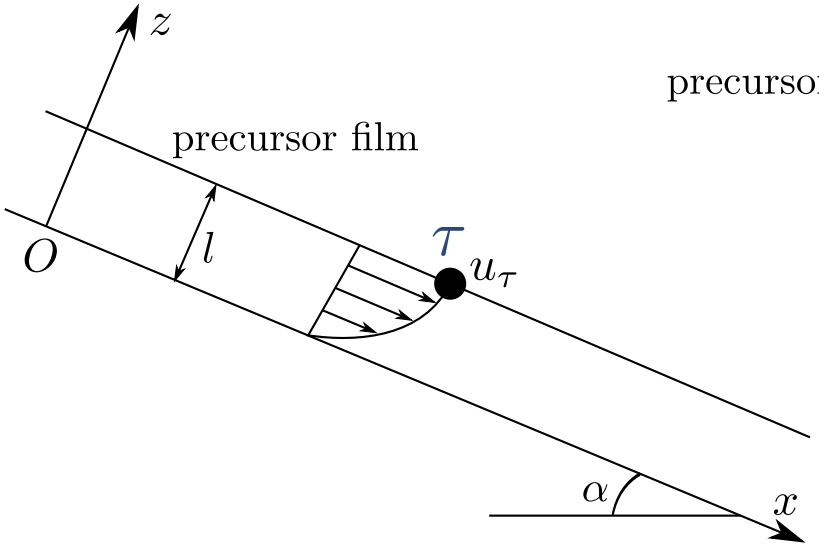
$$t - \frac{4}{15}\frac{A}{\beta^{15/4}}\left[\ln\left(\frac{B}{\beta^{3/4}}\right) - \frac{1}{5}\right] + C = 0 \tag{21}$$

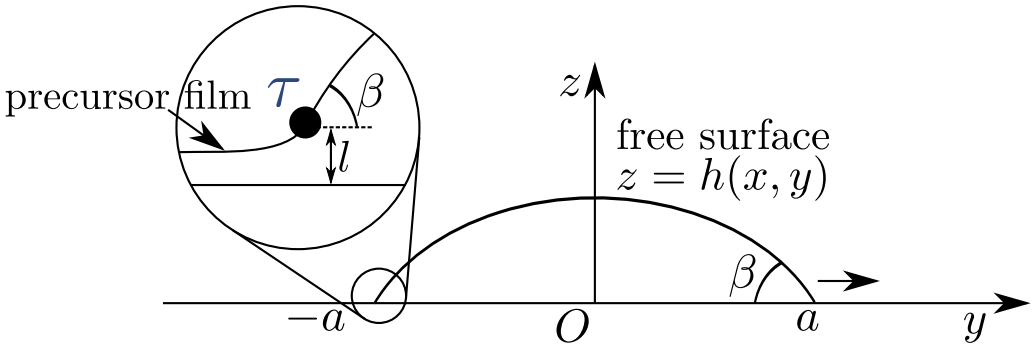
Introduction of initial condition, $\beta(0) = \beta_0$

$$C = \frac{4}{15}\frac{A}{\beta_0^{15/4}}\left[\ln\left(\frac{B}{\beta_0^{3/4}}\right) - \frac{1}{5}\right]$$

Transformation from t to x

Presence of falling thin liquid film, l, on whole plate





From t to x using u_τ

$$u_\tau = \frac{\rho g \sin\alpha}{2\mu} l^2 \implies t = \frac{2\mu}{\rho g l^2 \sin\alpha} x = \varpi x$$

Spreading of flowing rivulet along the plate

Substituting for t into the equation (21)

Implicit dependence of rivulet $(g)-(l)$ interface shape on x

$$x - \frac{\bar{A}}{\beta^{15/4}}\left[\ln\left(\frac{B}{\beta^{3/4}}\right) - \frac{1}{5}\right] + \bar{C} = 0 \tag{22}$$

$$\bar{A} = \frac{4}{15}\frac{A}{\varpi} \qquad \bar{C} = \frac{4}{15}\frac{C}{\varpi}$$

Notes on the equation (22)

- The obtained profiles will be all of the shape of circle segments.
- The problem of finding the shape of the rivulet's interface was reduced to specifying the right intermediate region length scale, l.

Spreading of flowing rivulet along the plate

Substituting for t into the equation (21)

Implicit dependence of rivulet $(g)-(l)$ interface shape on x

$$x - \frac{\bar{A}}{\beta^{15/4}}\left[\ln\left(\frac{B}{\beta^{3/4}}\right) - \frac{1}{5}\right] + \bar{C} = 0 \tag{22}$$

$$\bar{A} = \frac{4}{15}\frac{A}{\varpi} \qquad \bar{C} = \frac{4}{15}\frac{C}{\varpi}$$

Notes on the equation (22)

- The obtained profiles will be all of the shape of circle segments.
- The problem of finding the shape of the rivulet's interface was reduced to specifying the right intermediate region length scale, l.

Derived method for S_{g-l} calculation

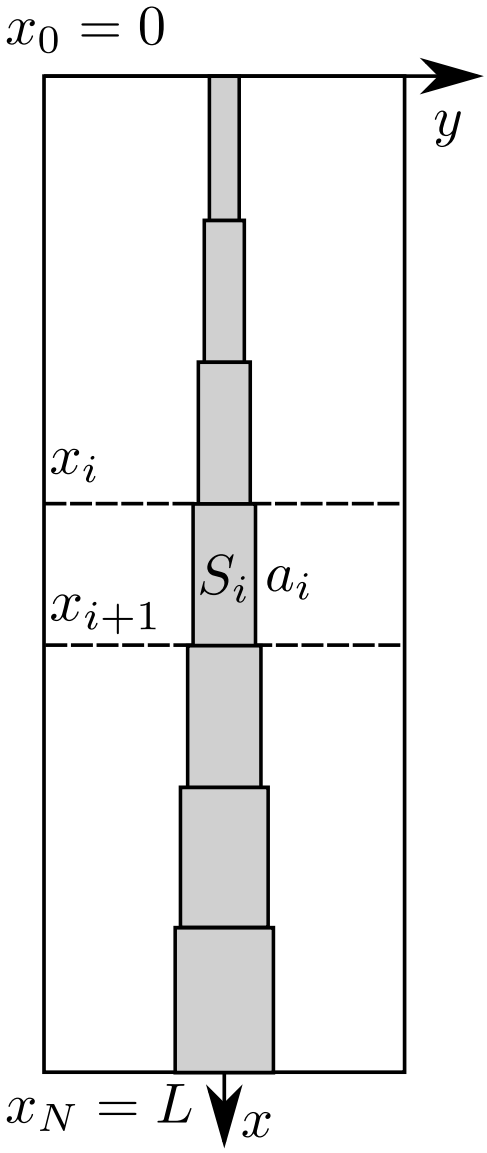
Equation (22) defines the shape of rivulet interface in implicit dependence of x

Proposed algorithm:

- Discretize domain in x to N subdomains
- For all subdomains solve (22) with $x = x_i$ and obtain β_i, $i = 1, 2, \ldots, N$
- From β_i calculate shape of each $(g) - (l)$ interface, $h_i(x, y)$
- Evaluate integral

$$S_{g-l} = \int_0^L \int_{-a(x)}^{a(x)} \sqrt{1 + \left(\frac{\partial h(x,y)}{\partial y}\right)^2} \, \mathrm{d}y \, \mathrm{d}x$$

and obtain S_{g-l}



8 Experiment

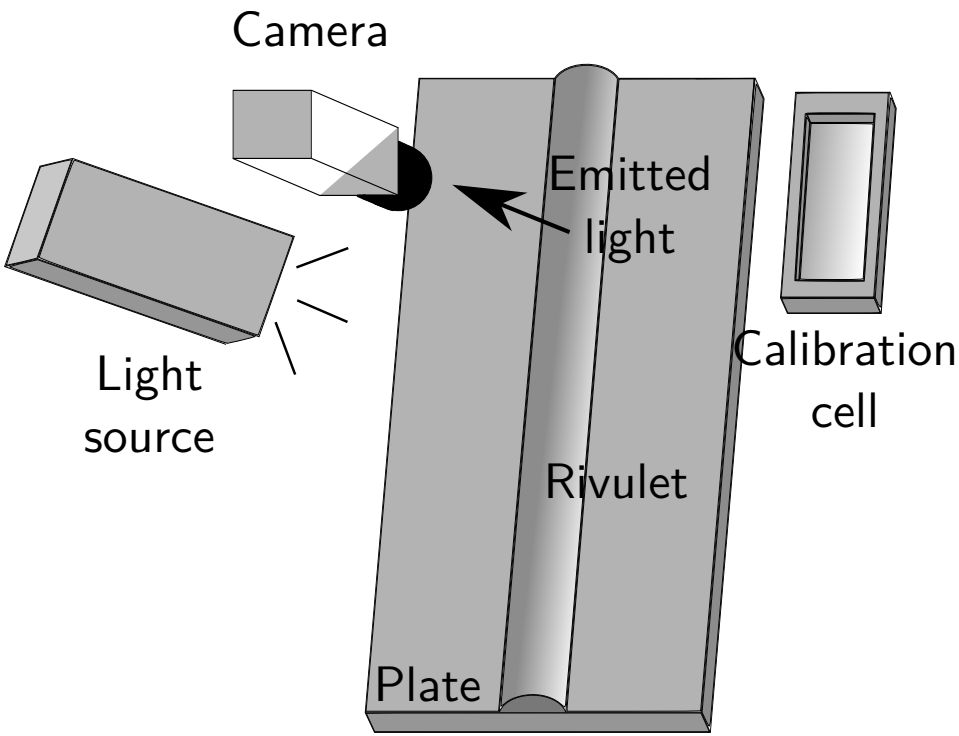
- Measurements principle
- Data Evaluation
- Data comparison

Measurements principle and data origin

Light Induced Fluorescence

Measurements principle – LIF[8, 9]

- Illumination of marked liquid by monochromatic light
- Measurements of emitted light intensities
- Conversion of measured light intensities in local film thicknesses

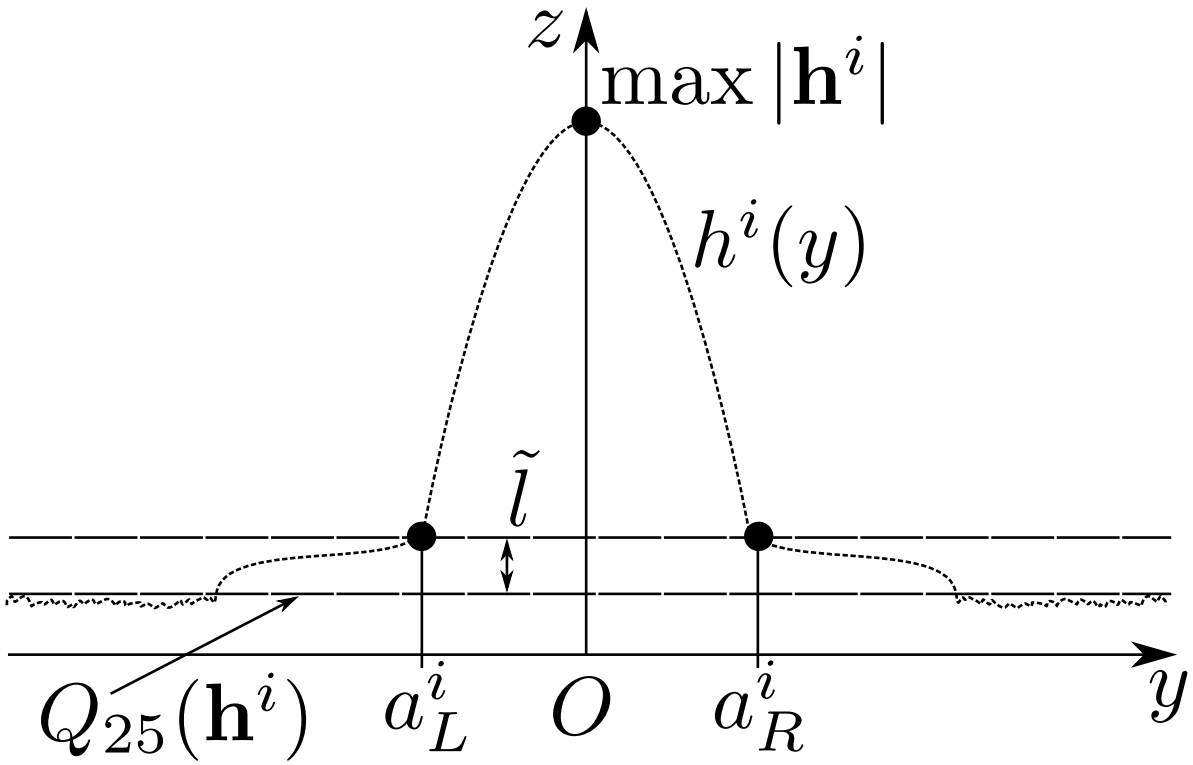


Output of measurements

Image of $(g) - (l)$ interface in a form of grayscale photography

Rivulet distinction

Find rivulet edges on the plate

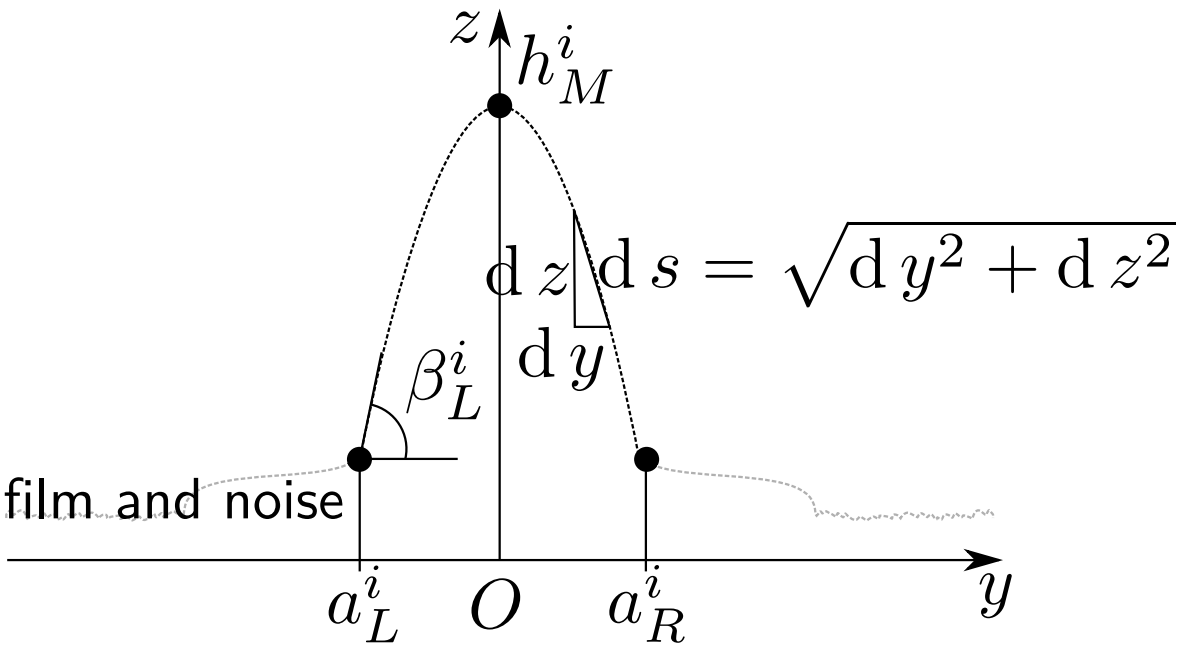


Rivulet edges identification

$a_L^i \iff$ first value of $\mathbf{h}^i < Q_{25}(\mathbf{h}^i) + \tilde{l}$ left from $\max|\mathbf{h}^i|$

Studied parameters evaluation

Pixel-wise calculation of S_{g-l} and of other rivulet type flow characteristics



$$S_{(g)-(l)} = \sum_{(i)} \left(\sum_{(j)} \mathrm{d}\,s_{ij} \right) \mathrm{d}x_i$$

Rivulet parameters

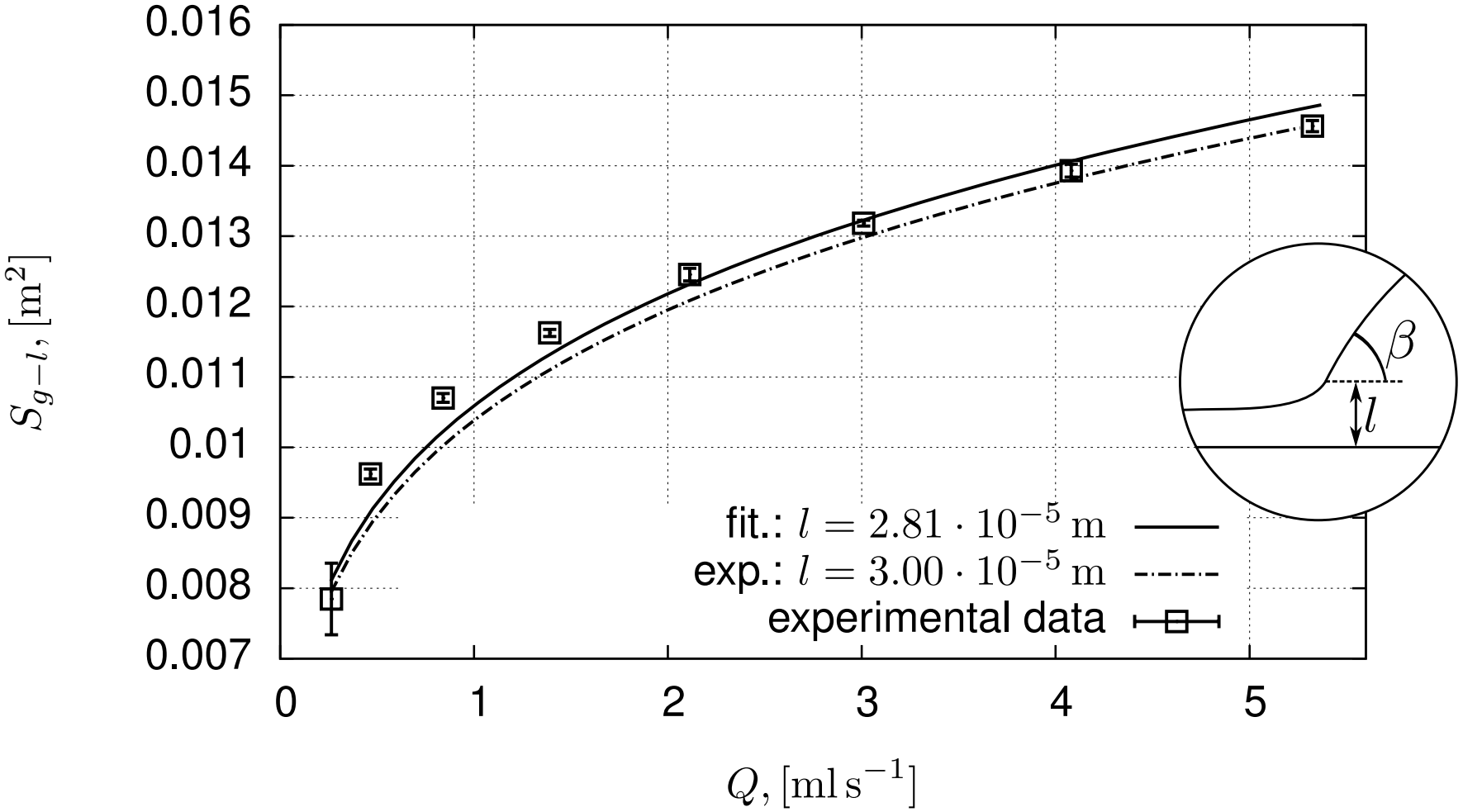
- S_{g-l}
- $(a_L^i + a_R^i)$, h_M^i
- β_L^i, β_R^i
- $\langle u \rangle_i$, Re_a^i, $\mathrm{Re}_{l_c}^i$

Number of observations

- All available transversal cuts for S_{g-l}
- 5 – 10 cuts for the rest

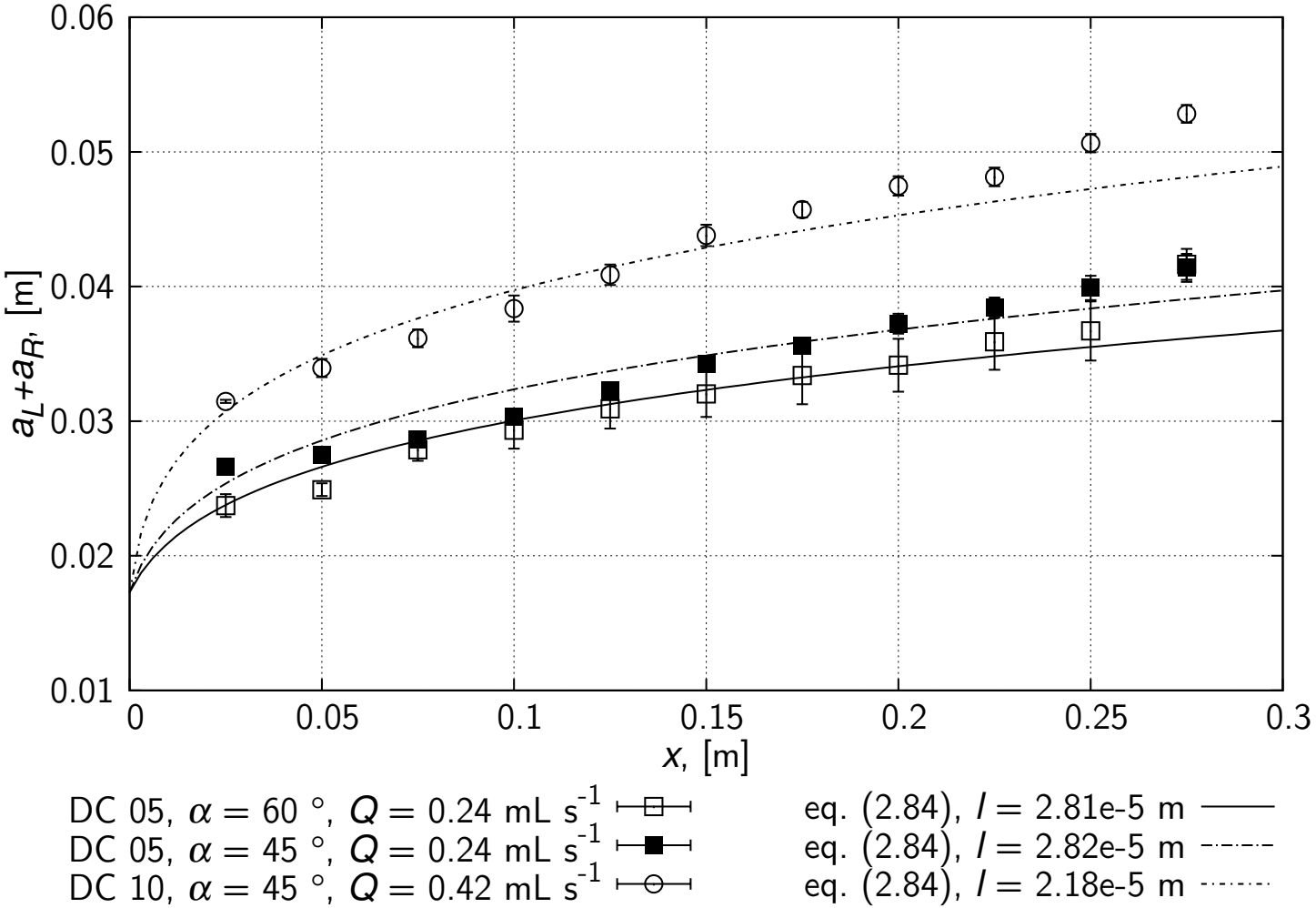
Comparison of calculated and measured S_{g-l}

Silicon oil spreading on steel, $\alpha = 60\,°$



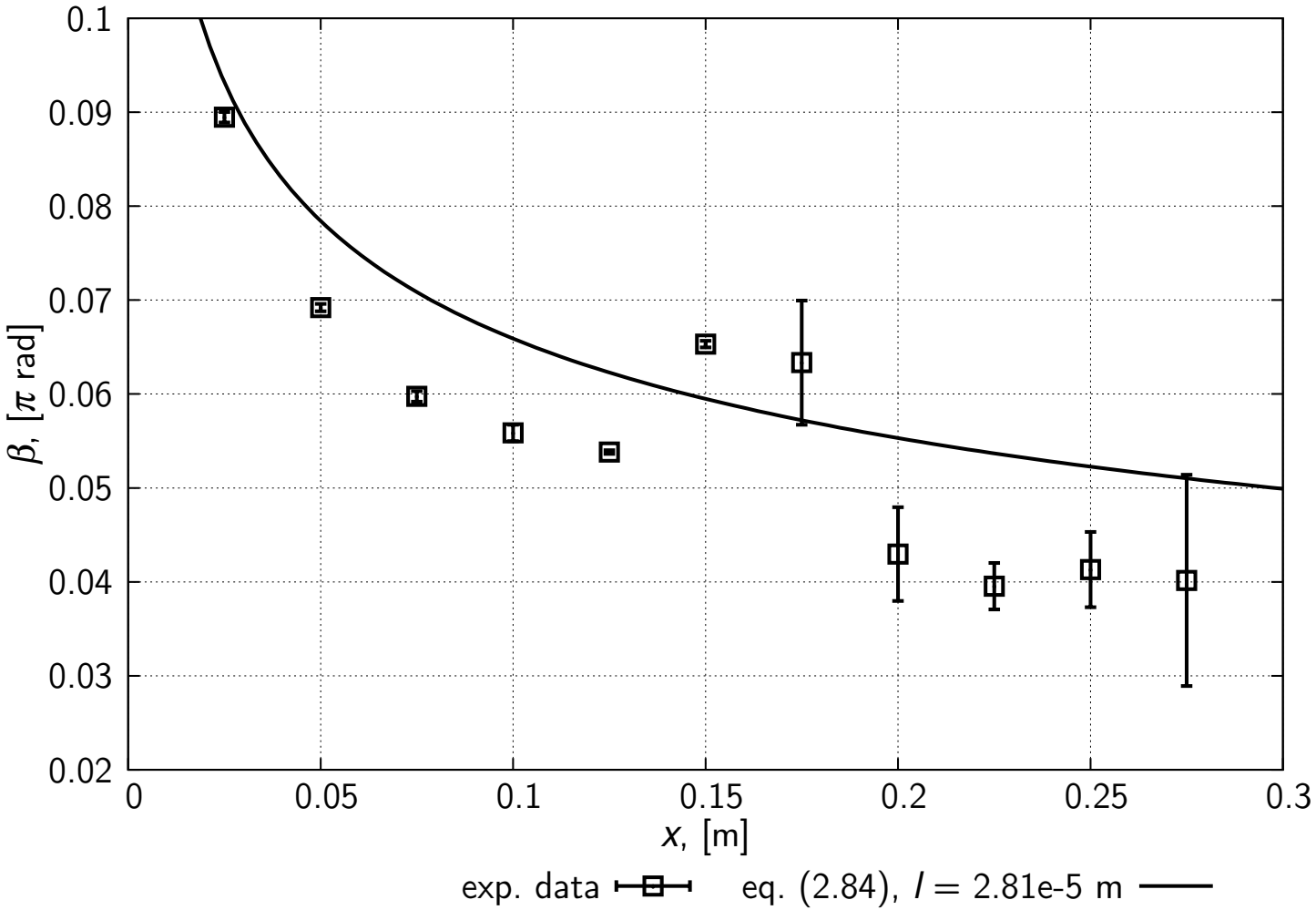
Comparison of calculated and measured $2a$

DC 05 a DC 10, various α



Comparison of calculated and measured β

DC 05, $\alpha = 45\,^\circ$, $Q = 4.90 \cdot 10^{-6}\,\mathrm{m}^3\,\mathrm{s}^{-1}$



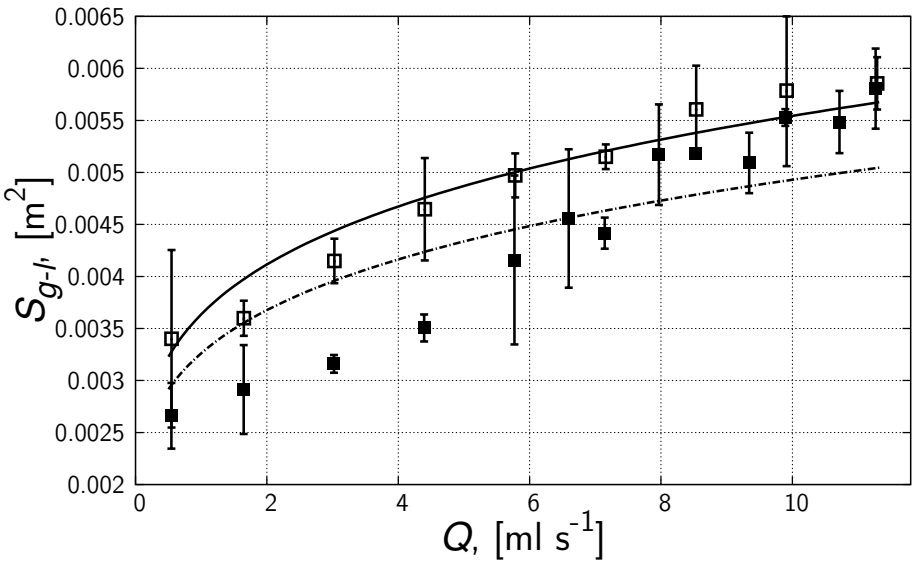
9 Conclusions

Outlook

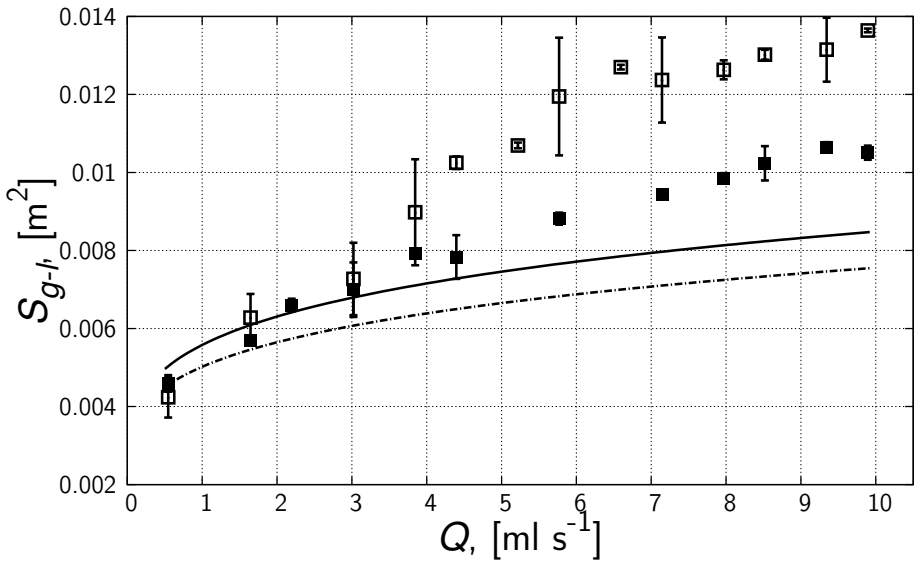
Dry plate, gravity, transitions from dry to wetted plate, waves?, applications?

Water

$l = 4.3 \cdot 10^{-5}$ m, $\Phi = 0.389$



eq. (2.84), $\alpha = 45$ ° —— eq. (2.84), $\alpha = 75$ ° -·-·-
exp. data, $\alpha = 45$ ° ⊟ exp. data, $\alpha = 75$ ° ■



eq. (2.84), $\alpha = 45$ ° —— eq. (2.84), $\alpha = 75$ ° -·-·-
exp. data, $\alpha = 45$ ° ⊟ exp. data, $\alpha = 75$ ° ■

Water with surfactants

$l = 3.3 \cdot 10^{-5}$ m, $\Phi = 0.624$

Outlook

Dry plate, gravity, transitions from dry to wetted plate, waves?, applications?

Dry plate

- Formation of precursor film $\implies \beta_m \neq 0$
- Surface tension acts **against** spreading

Gravity

- With gravity effects, shape of 2D $(g) - (l)$ interface is not an arc
- New term in thin liquid film governing equation

Transitions

- Critical presursor film height, l^*
- $l > l^*$: wetted plate
- $l < l^*$: dry plate

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In order of appearance

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In order of appearance

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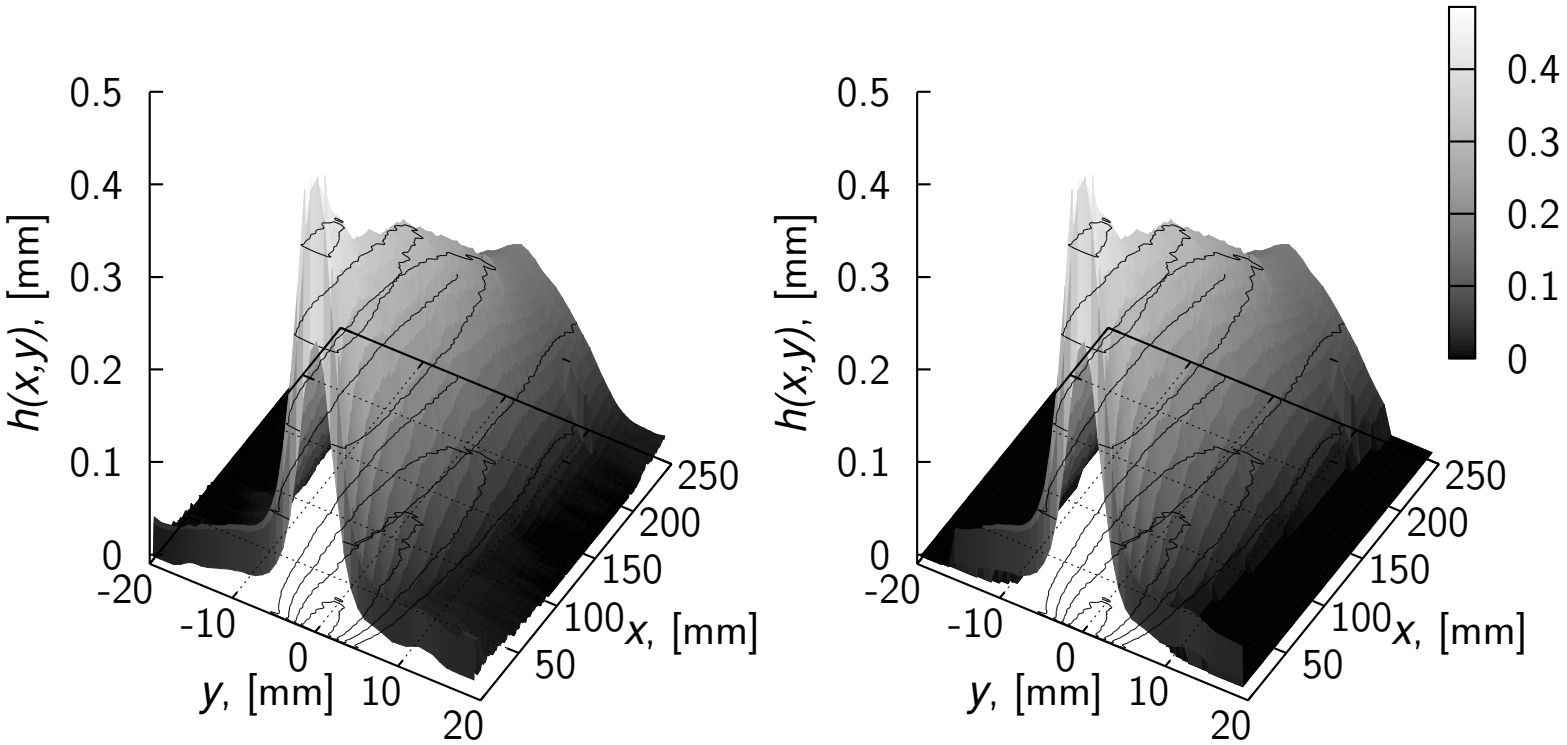
Thank you for your attention

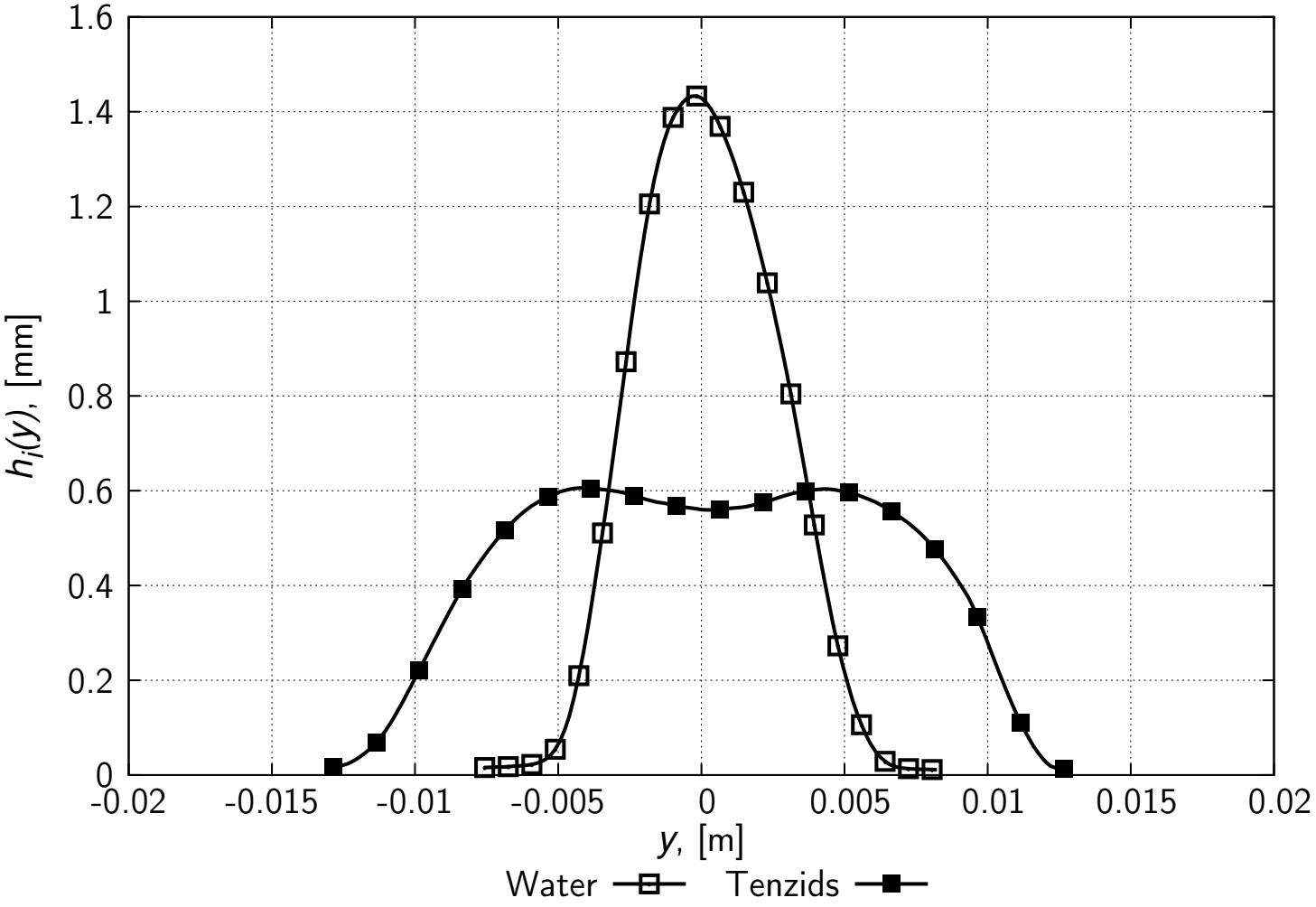
Accuracy of rivulet distinction

DC 10, $\alpha = 75\,^{\circ}$, $Q = 0.18 \cdot 10^{-6}\,\mathrm{m}^3\,\mathrm{s}^{-1}$



Influence of surfactants on liquids

$\alpha = 45°$, 15 cm from the plate top, $Q = 5.77 \cdot 10^{-6}\,\mathrm{m^3s^{-1}}$



Measured and simulated rivulet $(g) - (l)$ interface

DC 10, $\alpha = 52\,^{\circ}$, $Q = 0.42 \cdot 10^{-6}\,\mathrm{m}^3\,\mathrm{s}^{-1}$