POD-DEIM based model order reduction for speed-up of flow parametric studies

M. Isoz[a]

[a] Institute of Thermomechanics of the Czech Academy of Sciences

June 27, 2018

13th OpenFOAM Workshop 2018

Introduction

Original system

$$\dot{y} = Ay + f(t, y), \quad y(t) \in \mathbb{R}^m, \quad y(0) = y_0,\, t \in [0, T]\,,$$

system matrix ... $A \in \mathbb{R}^{m \times m}$,

nonlinearities ... $f(t, y) \in \mathbb{R}^m$

Reduced-order system

$$\dot{\eta}^\ell = A^\ell \eta^\ell + f^\ell(t, \eta^\ell), \quad \eta^\ell(t) \in \mathbb{R}^\ell, \quad \eta^\ell(0) = \eta_0^\ell,\, t \in [0, T],$$

system matrix ... $A^\ell \in \mathbb{R}^{\ell \times \ell}$,

nonlinearities ... $f^\ell(t, \eta^\ell) \in \mathbb{R}^\ell$

gain ... $\ell \ll m$

Proper orthogonal decomposition & Discrete empirical interpolation method

Introduce the Galerking ansatz and Fourier modes

- Prerequisitities:

$$\dot{y} = Ay + f(t, y), \quad y(t) \in \mathbb{R}^m, \quad y(0) = y_0,\, t \in [0, T]$$

$$y(t) \in V = \mathrm{span}\{\psi_j\}_{j=1}^{d} \quad \forall t \in [0, T]$$

$\Psi = \{\psi_j\}_{j=1}^{d}$ . . . orthonormal basis

$$y(t) = \sum_{j=1}^{d} \langle y(t), \psi_j \rangle_W \, \psi_j,\ \forall t \in [0, T], \quad W \text{ . . . appropriate weights}$$

- Ansatz for Galerkin projection, $\ell < d$

$$y^\ell(t) := \sum_{j=1}^{\ell} \langle y^\ell(t), \psi_j \rangle_W \, \psi_j,\ \forall t \in [0, T], \quad \eta_j^\ell(t) := \langle y^\ell(t), \psi_j \rangle_W$$

- Put the above together, !! $\psi_j \in \mathbb{R}^m$, $j = 1, \ldots, \ell$, $m > \ell$!!

$$\begin{aligned} \textstyle\sum_{j=1}^{\ell} \dot{\eta}_j^\ell \psi_j &= \textstyle\sum_{j=1}^{\ell} \eta_j^\ell A \psi_j + f(t, y^\ell(t)), \quad t \in (0, T) \\ y_0 &= \textstyle\sum_{j=1}^{\ell} \eta_j^\ell(0) \psi_j \end{aligned}$$

Introduce the reduced-order model

- Assume, that the above holds after projection on $V^\ell = \mathrm{span}\{\psi_j\}_{j=1}^{\ell}$, remember that $\langle \psi_j, \psi_i \rangle_W = \delta_{ij}$ and write,

$$\dot{\eta}_i^\ell = \sum_{j=1}^{\ell} \eta_j^\ell \langle A\psi_j, \psi_i \rangle_W + \langle f(t, y^\ell), \psi_i \rangle_W, \quad 1 \le i \le l \,\text{and}\, t \in (0, T]$$

- Define the matrix $A^\ell = (a_{ij}^\ell) \in \mathbb{R}^{l \times l}$ with $a_{ij}^\ell = \langle A\psi_j, \psi_i \rangle_W$
- Define the vector valued mapping $\eta^\ell = (\eta_1^\ell, \ldots, \eta_l^\ell)^{\mathrm{T}} : [0, T] \to \mathbb{R}^\ell$
- Define the non-linearity $f^\ell = (f_1^\ell, \ldots, f_l^\ell)^{\mathrm{T}} : [0, T] \to \mathbb{R}^\ell$, where

$$f_i^\ell(t, \eta) = \left\langle f \left(t, \sum_{j=1}^{\ell} \eta_j \psi_j \right), \psi_i \right\rangle_W$$

- Introduce the IC, $\eta^\ell(0) = \eta_0^\ell = (\langle y_0, \psi_1 \rangle_W, \ldots, \langle y_0, \psi_1 \rangle_W)^{\mathrm{T}}$
- Write the ROM, $\dot{\eta}^\ell = A^\ell \eta^\ell + f^\ell(t, \eta^\ell)$, for $t \in (0, T]$, $\eta^\ell(0) = \eta_0^\ell$

Original system

$$\dot{y} = Ay + f(t, y), \quad y(t) \in \mathbb{R}^m, \quad y(0) = y_0,\, t \in [0, T],$$

Solution snapshots←Approximation obtained from FOM

$$\boldsymbol{S} = \left\{ \boldsymbol{y}_j = \boldsymbol{y}(t_j) = \mathrm{e}^{At_j}\boldsymbol{y}_0 + \int_0^{t_j} \mathrm{e}^{A(t_j - s)}\boldsymbol{b}(s, \boldsymbol{y}(s))\,\mathrm{d}s \right\}_{j=1}^{n} \approx \tilde{\boldsymbol{S}} \leftarrow \text{FOM}$$

Matrix of snapshots (tildes denoting approximate solutions are omitted)

$$Y = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n] \in \mathbb{R}^{m\times n}, \quad \mathrm{rank}(Y) = d \leq \min\{m, n\}\,,$$

Goal

Approximate all the spatial coordinate vectors $\boldsymbol{y}_j$ of Y simultaneously by $\ell \leq d$ normalized vectors as well as possible.

(P)

$$\boxed{\begin{gathered}\max_{\tilde{\boldsymbol{\psi}}_1,\ldots,\tilde{\boldsymbol{\psi}}_\ell \in \mathbb{R}^m} \sum_{i=1}^{\ell} \sum_{j=1}^{n} \left| \langle \boldsymbol{y}_j, \tilde{\boldsymbol{\psi}}_i \rangle_{\mathbb{R}^m} \right|^2 \\ \text{subject to} \\ \langle \tilde{\boldsymbol{\psi}}_i, \tilde{\boldsymbol{\psi}}_j \rangle_{\mathbb{R}^m} = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq \ell\,,\end{gathered}}$$

Fundamental theorem of Proper orthogonal decomposition

Let Y be a given matrix of snapshots. Also, let $Y = \Psi\Sigma\Phi^T$ be the singular value decomposition of Y, where $\Psi = [\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_m] \in \mathbb{R}^{m\times m}$ and $\Phi = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n] \in \mathbb{R}^{n\times n}$ are orthogonal matrices and the matrix Σ has the structure of

$$\Sigma = \begin{bmatrix} \mathrm{diag}(\sigma_1, \ldots, \sigma_d) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{m\times n},$$

where $\sigma_1, \ldots, \sigma_d$ are the singular values of the matrix Y. Then, for any $\ell \in \{1, \ldots, d\}$ the solution to problem (**P**) is given by the singular vectors $\{\boldsymbol{\psi}_i\}_{i=1}^{\ell}$, i.e. by the first ℓ columns of Ψ. Moreover,

$$\mathrm{argmax}(\mathbf{P}) = \sum_{i=1}^{\ell} \sigma^2 .$$

Proof

- Obtained via Lagrange framework
- Rather long and technical, can be found in literature (e.g. [**VolkweinBook**])

Deal with the non-linearities I

- Identify the problem,

$$f_i^\ell(t,\eta) = \left\langle f\left(t, \sum_{j=1}^{\ell} \eta_j \psi_j\right), \psi_i \right\rangle_W \quad \ldots \sum_{j=1}^{\ell} \eta_j \psi_j \in \mathbb{R}^m \leftarrow \text{FO}$$

- Approximate the non-linearities via the POD basis, Φ,

$$b(t) := f(t, \Psi\eta^\ell) \approx \sum_{k=1}^{p} \phi_k c_k(t) = \Phi c(t) \ \ldots \text{ Galerkin ansatz}$$

- Approximate $f^\ell(t,\eta^\ell)$ through Ψ, W, Φ,

$$f^\ell(t,\eta^\ell) = \Psi^{\mathrm{T}} W f(t, \Psi\eta^\ell) = \Psi^{\mathrm{T}} W b(t) \approx \Psi^{\mathrm{T}} W \Phi c(t), \quad c(t) \in \mathbb{R}^p$$

- Select (greedy) only p rows to make system consistent, introduce $\vec{i}$

$$P := [e_{\vec{i}1}, \ldots, e_{\vec{i}p}] \in \mathbb{R}^{m\times p},\ e_{\vec{i}k} = (0,\ldots,0,1,0,\ldots,0)^{\mathrm{T}} \in \mathbb{R}^m$$

Deal with the non-linearities II (yes, almost done)

- Plug in the matrix P,

$$P^{\mathrm{T}}\Phi c(t) \approx P^{\mathrm{T}} b(t), \leftarrow c(t) \in \mathbb{R}^p,\ \Phi \in \mathbb{R}^{m\times p},\ b(t) \in \mathbb{R}^m$$

$$\det(P^{\mathrm{T}}\Phi) \neq 0 \implies c(t) \approx (P^{\mathrm{T}}\Phi)^{-1} P^{\mathrm{T}} b(t) = (P^{\mathrm{T}}\Phi)^{-1} P^{\mathrm{T}} f(t, \Psi\eta^{\ell})$$

- If $f(t, \Psi\eta^{\ell})$ is pointwise evaluable,

$$(P^{\mathrm{T}}\Phi)^{-1} P^{\mathrm{T}} f(t, \Psi\eta^{\ell}) = (P^{\mathrm{T}}\Phi)^{-1} f(t, P^{\mathrm{T}}\Psi\eta^{\ell}), \quad P^{\mathrm{T}}\Psi\eta^{\ell} \in \mathbb{R}^p$$

- Write the final ROM

$$\dot{\eta}^{\ell} = A^{\ell}\eta^{\ell} + f^{\ell}(t, \eta^{\ell}), \text{ for } t \in (0, T], \quad \eta^{\ell}(0) = \eta_0^{\ell},$$

where

$$f^{\ell}(t, \eta^{\ell}) = \Psi^{\mathrm{T}} W \Phi (P^{\mathrm{T}}\Phi)^{-1} f(t, P^{\mathrm{T}}\Psi\eta^{\ell})$$

Link with OpenFOAM

FVM semi-discretized PDR

$$\boxed{\Delta\Omega^h \dot{y} + \mathcal{L}^h(t,y) = 0 \implies \dot{y} = -(\Delta\Omega^h)^{-1}\mathcal{L}^h(t,y),}$$

$\mathcal{L}^h = -\tilde{A}(t)y - \tilde{b}(t,y)$ ... FVM spatial discretization operator in OpenFOAM

After some operations

$$\dot{y} = A(t)y + b(t,y), \quad A(t) = (\Delta\Omega^h)^{-1}\tilde{A}(t),\ b(t,y) = (\Delta\Omega^h)^{-1}\tilde{b}(t,y)$$

Needed snapshots

$$\boxed{\boldsymbol{S} = \{\boldsymbol{y}_j, A_i, b_i)\}_{i=1}^{n}, \quad A_i \in \mathbb{R}^{m \times m}}$$

A_i ...sparse, $\sim 7m$ non-zero elements $\implies$ $\sim 7m$ floats and $\sim 8m$ integers to be stored

Linear interpolation between the stored matrices

$$\varpi(t) = \frac{t - t_{i-1}}{t_i - t_{i-1}}, \quad \hat{A}(t) = \varpi(t)A_{i-1} + (1 - \varpi(t))A_i$$

$$\boxed{\begin{aligned} \hat{A}^{\ell}(t) &= \Psi^{\mathrm{T}} W \hat{A}(t)\Psi = \Psi^{\mathrm{T}} W \left(\varpi(t)A_{i-1} + (1 - \varpi(t))A_i\right)\Psi = \\ &= \varpi(t)\Psi^{\mathrm{T}} W A_{i-1}\Psi + (1 - \varpi(t))\Psi^{\mathrm{T}} W A_i \Psi = \varpi(t)A^{\ell}_{i-1} + (1 - \varpi(t))A^{\ell}_i \end{aligned}}$$

Parameter dependent steady state, $\mu \in \mathbb{R}$

$$\begin{aligned} \mathcal{M}_x := \nabla \cdot (u \otimes u) - \nabla \cdot (\nu \nabla u) &= -\nabla \tilde{p} + \tilde{f} \\ \nabla \cdot u &= 0 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} BD^{-1}B^{\mathrm{T}}\tilde{p} = BD^{-1}\left(f - (L+U)u^*\right) \\ u = D^{-1}\left(f - (L+U)u^*\right) - D^{-1}B^{\mathrm{T}}p \end{aligned}$$

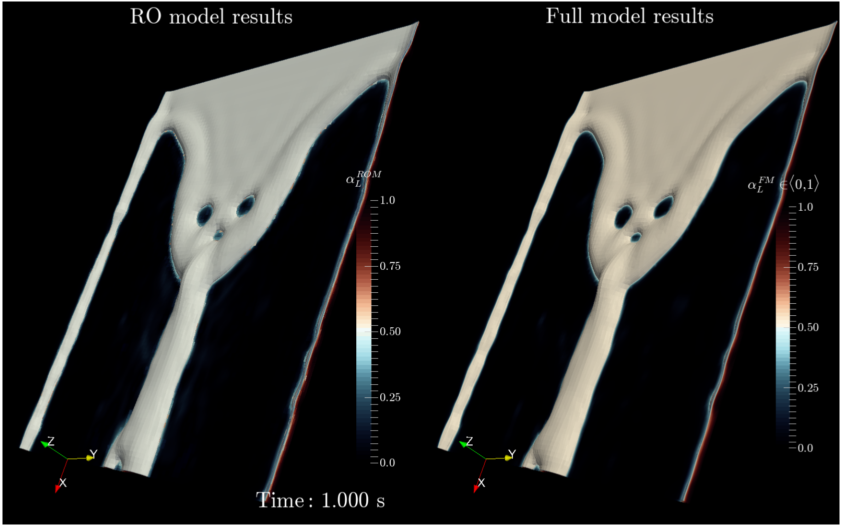
$$\boxed{0 = A(\mu)p + b(\mu, p), \quad A(\mu) := BD^{-1}B^{\mathrm{T}},\ b(\mu, p) := -BD^{-1}\left(f - (L+U)u^*\right)}$$

Transient case

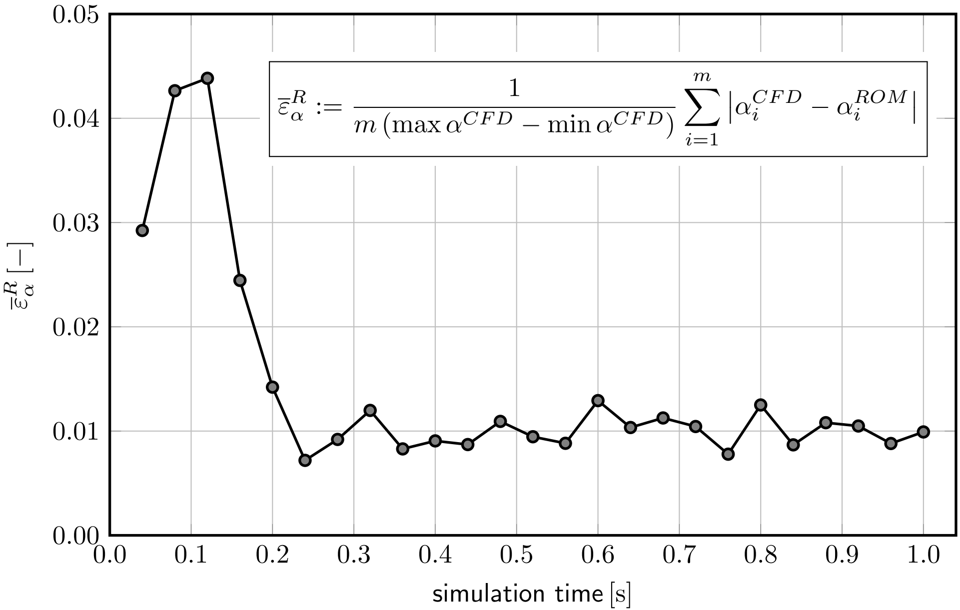
$$\mathcal{M}_t := u_t, \quad \mathcal{M}_t + \mathcal{M}_x = -\nabla \tilde{p} + \tilde{f}$$

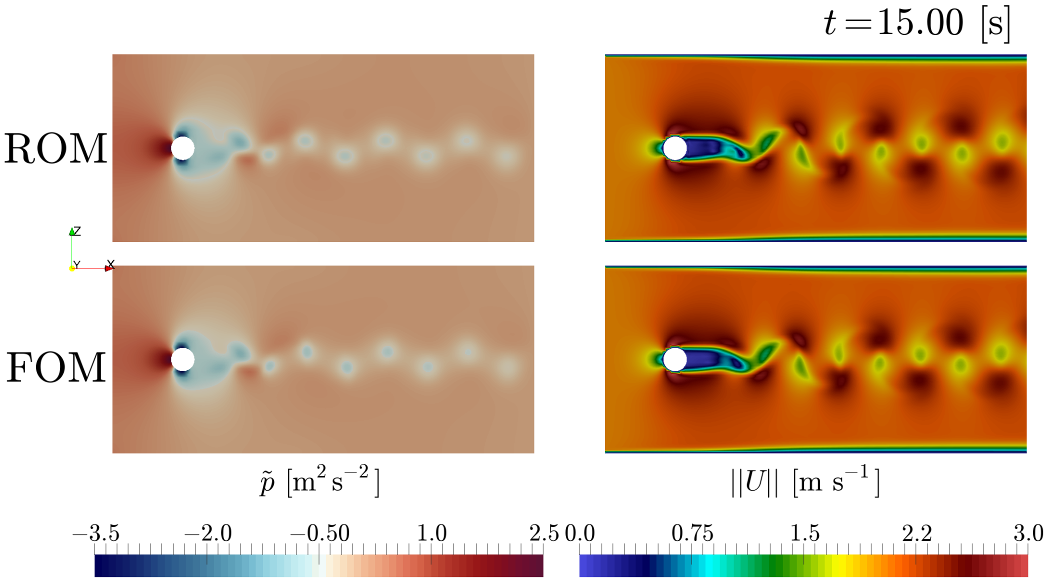
use only $\mathcal{M}_x$ to construct the pressure equation

$$\boxed{\text{"}\dot{p}\text{"} = A(t)p + b(t, p), \quad A(t) := BD^{-1}B^{\mathrm{T}},\ b(t, p) := -BD^{-1}\left(f - (L+U)u^*\right)}$$

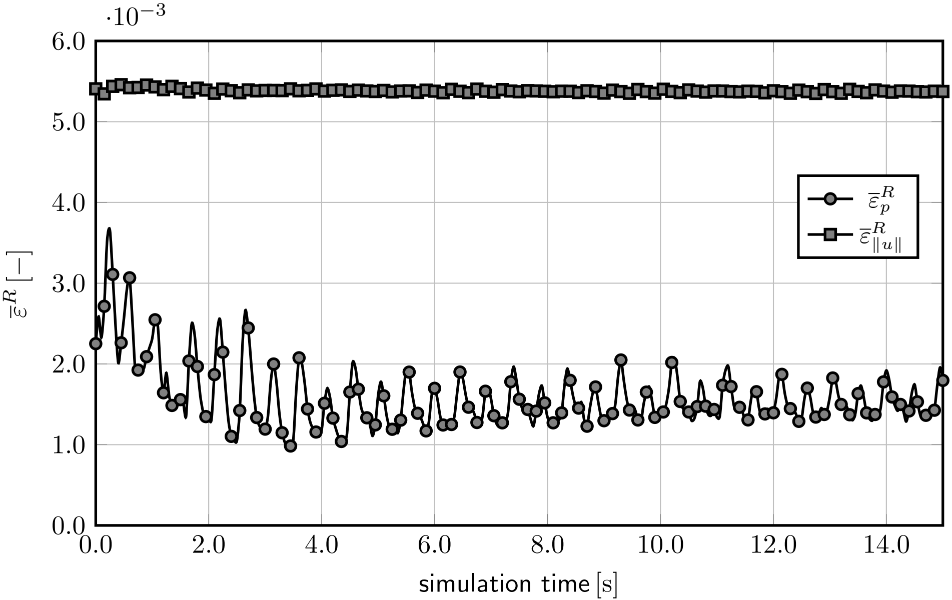


RO model results
Full model results
α_L^{ROM}
$\alpha_L^{FM} \in \langle 0,1 \rangle$
1.0
0.75
0.50
0.25
0.0
Z
Y
X
Time : 1.000 s

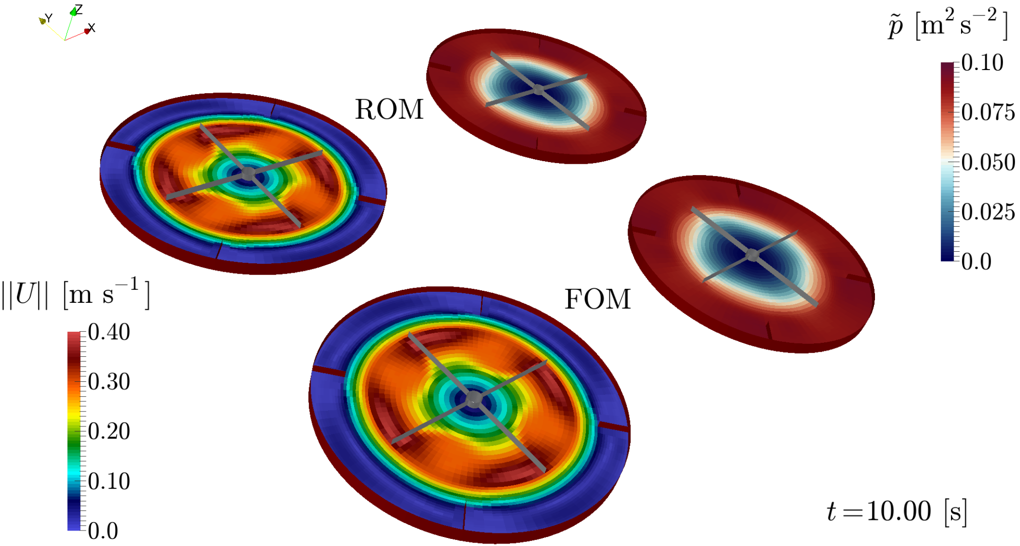

$$\overline{\varepsilon}_\alpha^R := \frac{1}{m\left(\max \alpha^{CFD} - \min \alpha^{CFD}\right)} \sum_{i=1}^{m} \left|\alpha_i^{CFD} - \alpha_i^{ROM}\right|$$
0.05
0.04
0.03
0.02
0.01
0.00
$\overline{\varepsilon}_\alpha^R$ [−]
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
simulation time [s]



$t = 15.00$ [s]
ROM
FOM
Z
Y
X
$\tilde{p}$ [m^2 s^{-2}]
$||U||$ [m s^{-1}]
−3.5
−2.0
−0.50
1.0
2.5
0.0
0.75
1.5
2.2
3.0



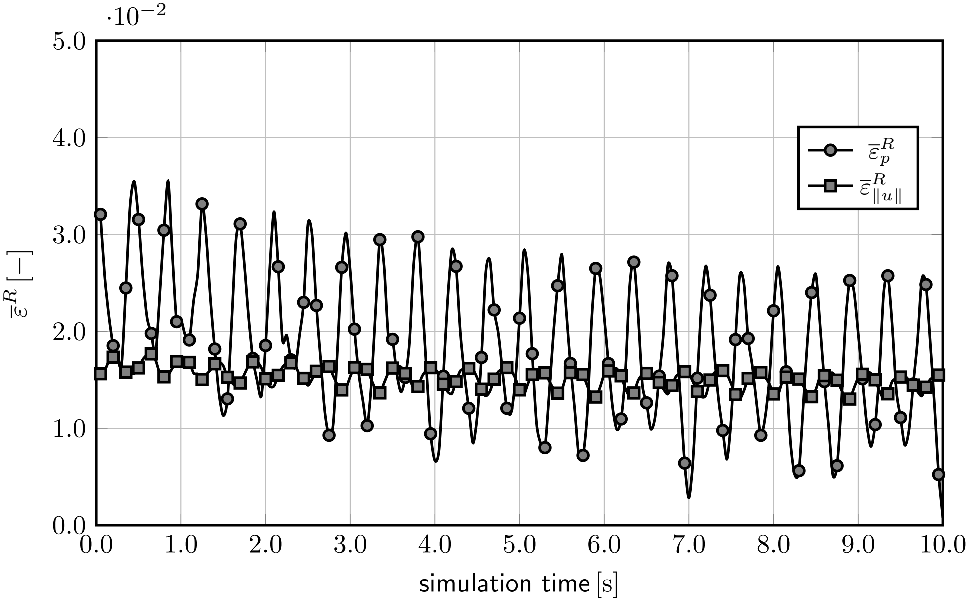
$\cdot 10^{-3}$
6.0
5.0
4.0
3.0
2.0
1.0
0.0
$\overline{\varepsilon}^R\ [-]$
0.0
2.0
4.0
6.0
8.0
10.0
12.0
14.0
simulation time [s]
$\overline{\varepsilon}_p^R$
$\overline{\varepsilon}_{\|u\|}^R$

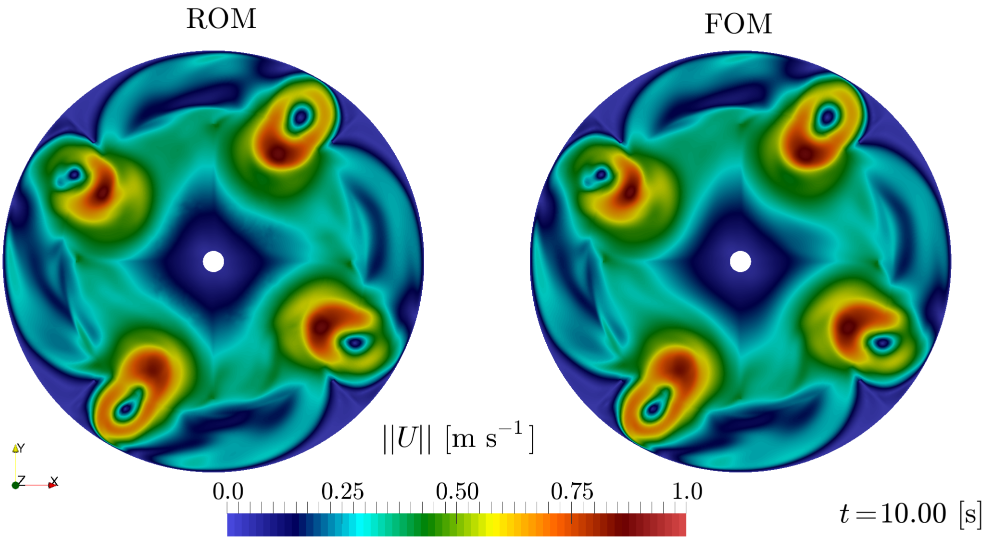


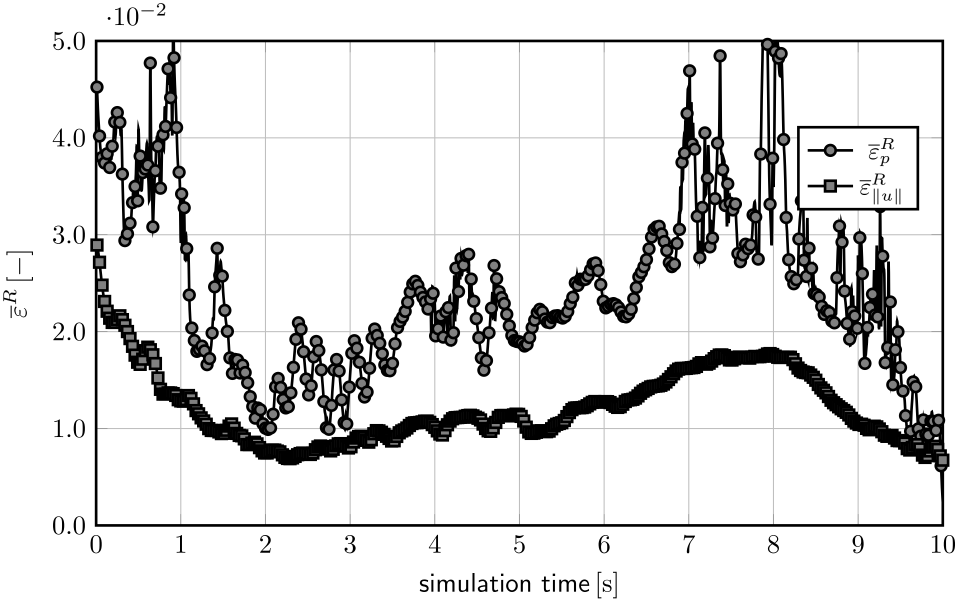
Y
Z
X
ROM
FOM
$\tilde{p}$ [m^2 s^{-2}]
0.10
0.075
0.050
0.025
0.0
$||U||$ [m s^{-1}]
0.40
0.30
0.20
0.10
0.0
$t = 10.00$ [s]

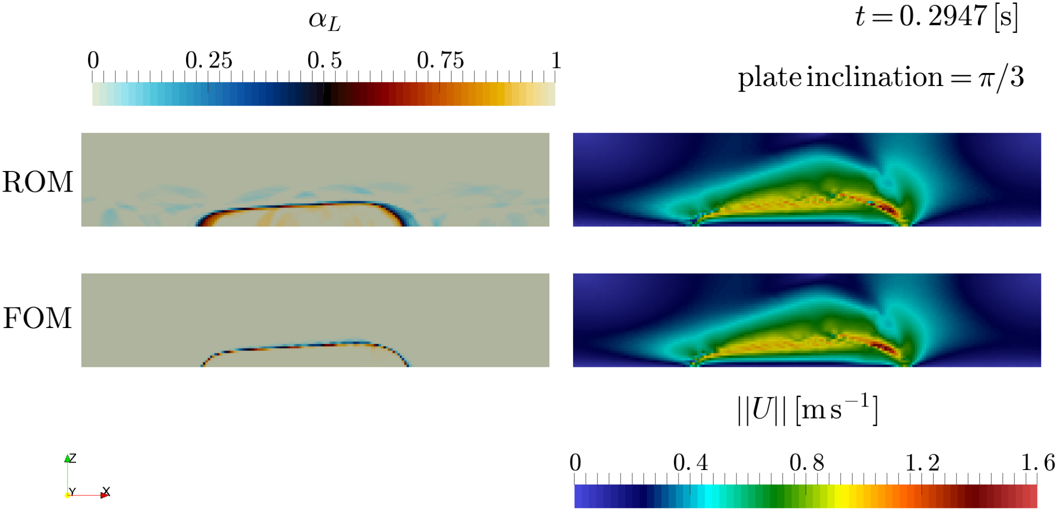
Example 3 – 2D mixer

Validation of the approach – arbitrary mesh interface

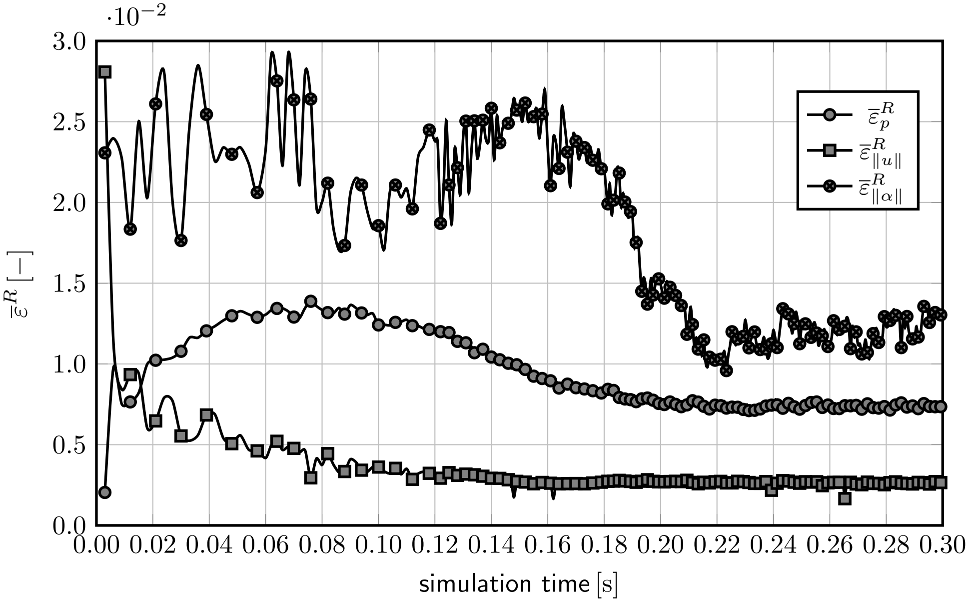



ROM
FOM
$||U||$ [m s^{-1}]
0.0
0.25
0.50
0.75
1.0
$t = 10.00$ [s]
Y
Z
X


$\cdot 10^{-2}$
5.0
4.0
3.0
2.0
1.0
0.0
$\overline{\varepsilon}^R$ [–]
0
1
2
3
4
5
6
7
8
9
10
simulation time [s]
$\overline{\varepsilon}_p^R$
$\overline{\varepsilon}_{\|u\|}^R$



α_L
0
0.25
0.5
0.75
1
$t = 0.2947$ [s]
plate inclination $= \pi/3$
ROM
FOM
$||U||$ [m s^{-1}]
0
0.4
0.8
1.2
1.6
Z
Y
X


$\cdot 10^{-2}$
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14 0.16 0.18 0.20 0.22 0.24 0.26 0.28 0.30
simulation time [s]
$\overline{\varepsilon}^R$ [–]
$\overline{\varepsilon}_p^R$
$\overline{\varepsilon}_{\|u\|}^R$
$\overline{\varepsilon}_{\|\alpha\|}^R$

Applications

Suitable snapshot selection: Leverage the continuity equation

Define cost function for snapshot selection

$$S_{\mathrm{cont}} = \sum_{\Omega_P^h \in \Omega^h} \left| \sum_{f \in \{f\}_P} \Phi_f^V \right|, \qquad S_{\mathrm{cont}}^{\mathrm{nn}} = d_{\mathrm{nn}}^q S_{\mathrm{cont}}$$

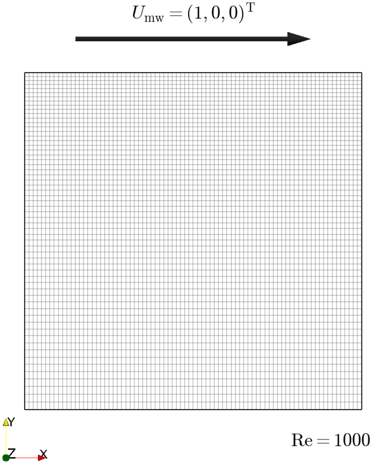
Identify snapshot to include into the basis

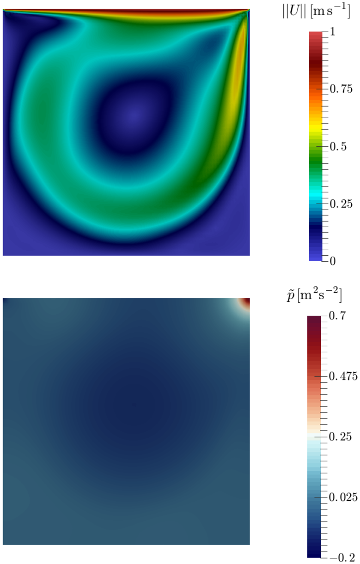
$(\mathbf{P}_{\tilde{\mu}}^{\mathrm{nn}})$

$$\boxed{\tilde{\mu} := \operatorname*{argmax}_{\mu \in D \setminus \boldsymbol{\mu}_{\mathrm{u}}} S_{\mathrm{cont}}^{\mathrm{nn}}(\mu)}$$

Use ROM to speed up parametric studies

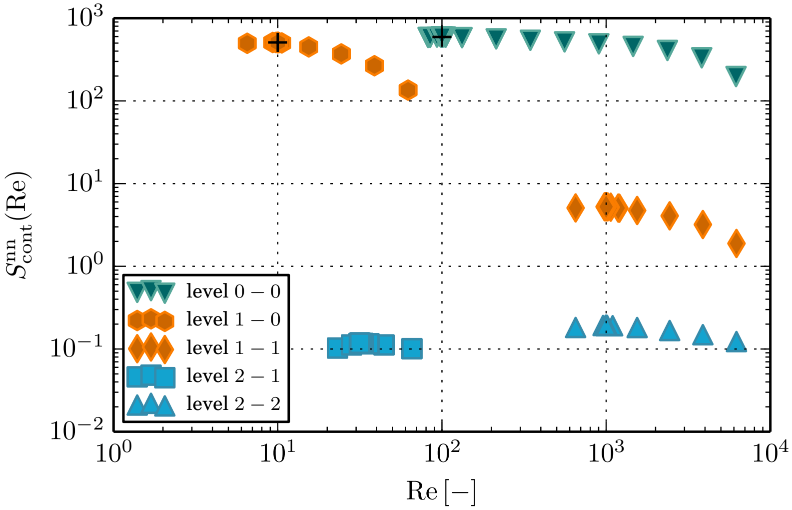
Sample parameter space and predict initial guesses for SIMPLE





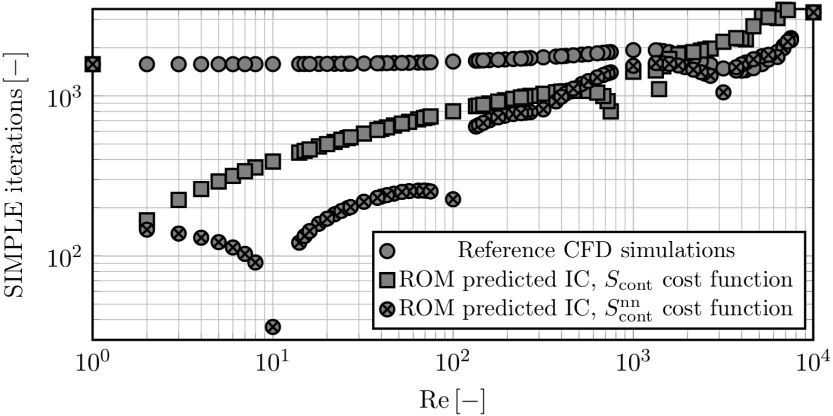
Use ROM to speed up parametric studies

Sample parameter space and predict initial guesses for SIMPLE



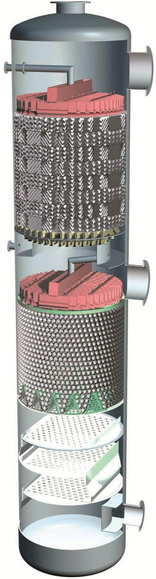
Use ROM to speed up parametric studies

Sample parameter space and predict initial guesses for SIMPLE



Real-life applications

ROM is a tremendous tool for parametric studies or repeated model evaluations



[Sulzer ChemTech]

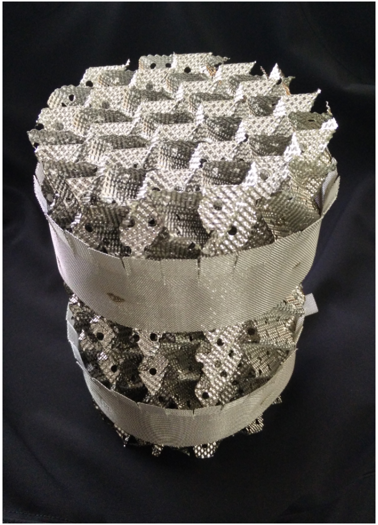
Importance

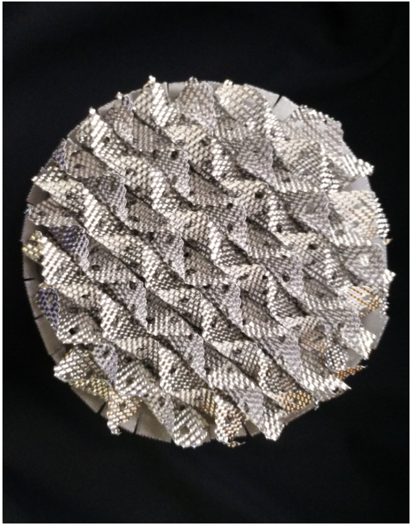
- Chemical industry creates mixtures but sells "pure species" (e.g. oil)
- 2014, 3% of energy consumption of the USA was due to the separation columns

Challenges

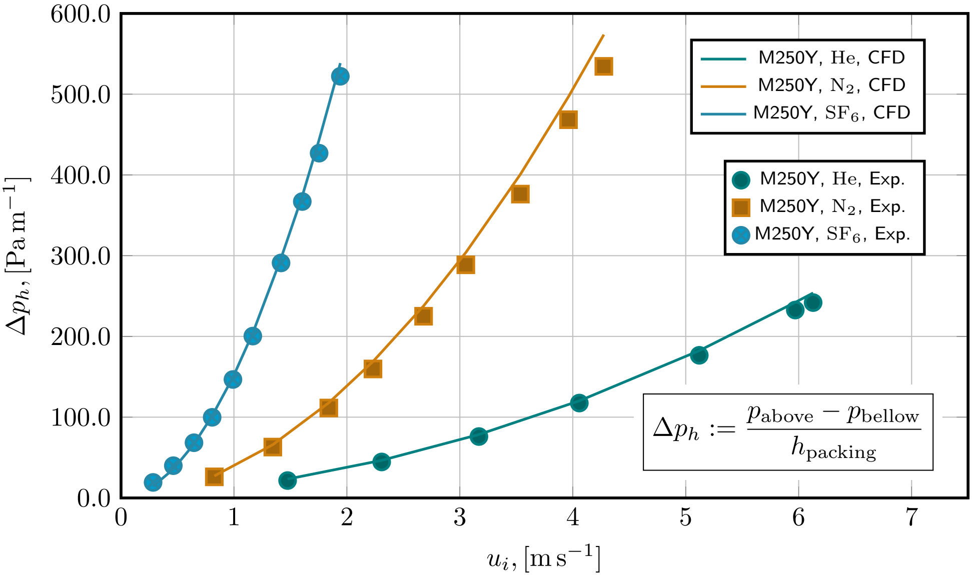
- Multiphase flow → non-steady process
- Complex geometry
- Simultaneous heat and mass transfer

Challenge: Geometry of structured packing

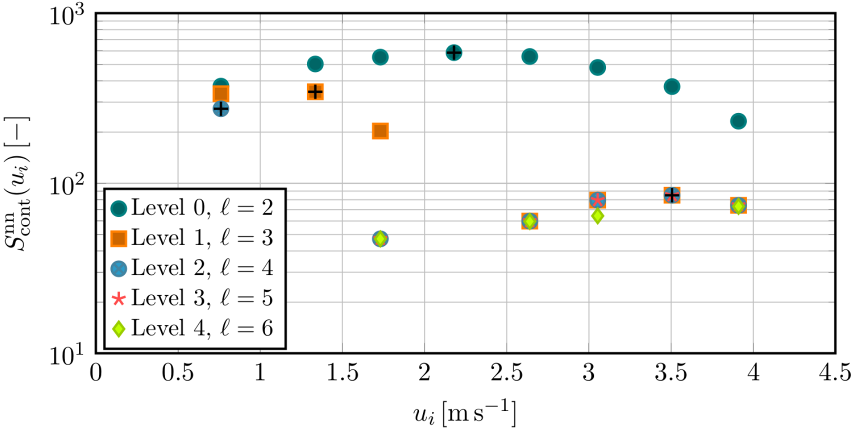




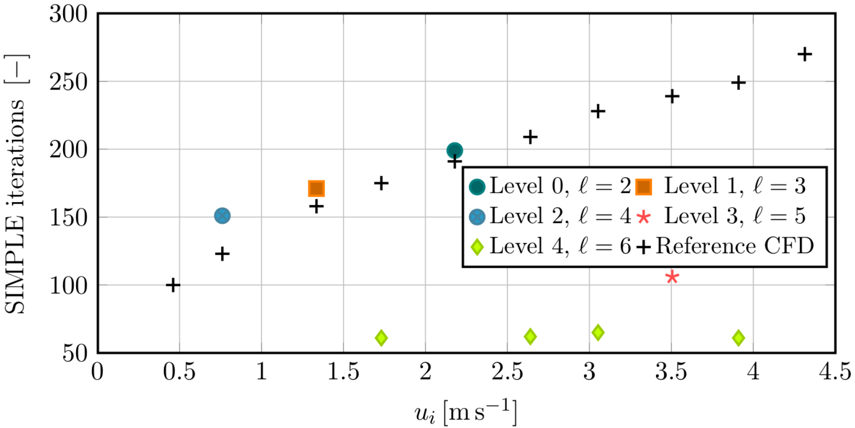
Comparison with experimental data: [Haidl, J. UCT Prague]



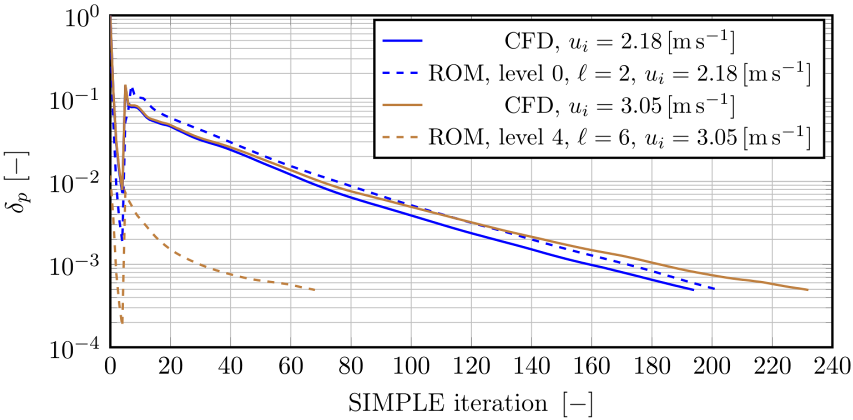
Snapshot selection:



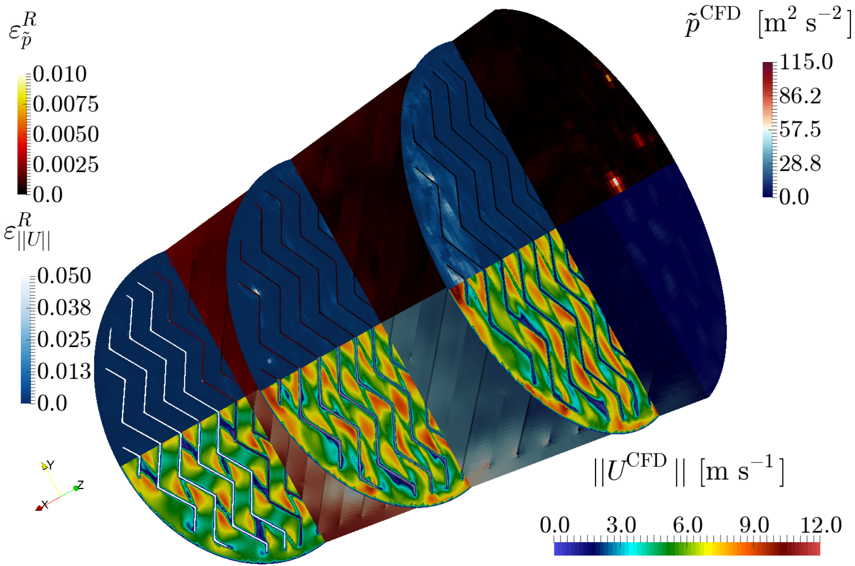
SIMPLE iterations:



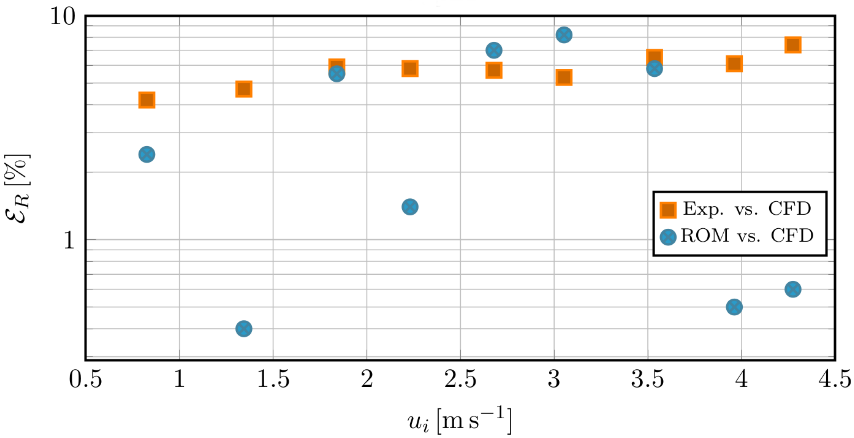
Residuals evolution:



Full case: Predicted vs. converged solution, $\ell = 6\,\mathrm{N_2}$ gas



Comparison with experimental data: $\ell = 6$, N_2 gas



Cost function: Single phase, toy problem

$$F(u_0) = \frac{\Delta\tilde{p} - \Delta\tilde{p}_{Max}}{\Delta\tilde{p}_{Max}} + K\frac{Q^2 - 2Q_{Max}Q + Q_{Min}(2Q_{Max} - Q_{Min})}{(Q_{Max} - Q_{Min})^2},$$

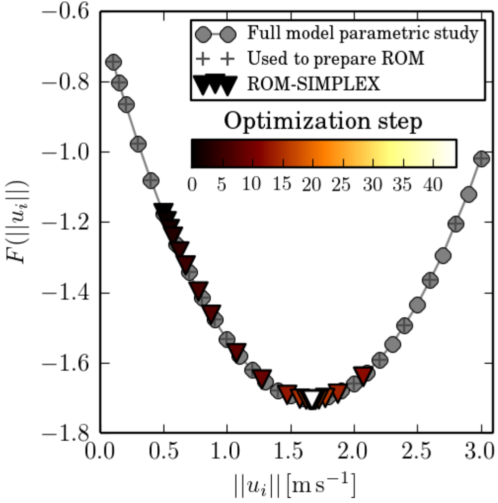
$$\Delta\tilde{p} = \Delta\tilde{p}(u_0), \quad Q = Q(u_0), \quad U_0 = (-u_0, 0, 0),$$

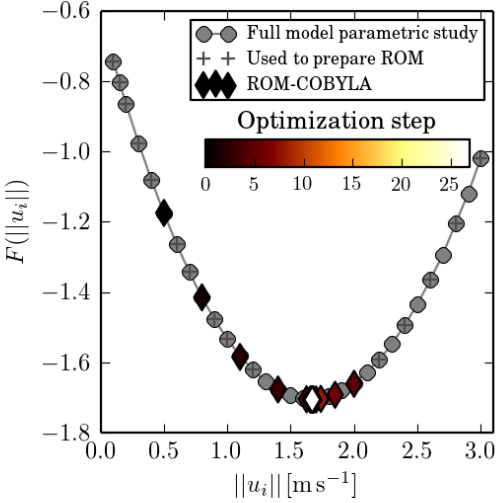
$\Delta\tilde{p}_{Max}$ .... maximal allowable pressure loss

Q_{Max}, (Q_{Min}) .... maximal, (minimal) allowable gas flow rate

K .... relative importance of the two terms

Cost function minimization: Results of SIMPLEX and COBYLA algorithms





Conclusions

Currently available

- Extended snapshot preparation for `simpleFoam`, `pimpleFoam` and `interFoam`
- Python module for ROM creation based on prepared outputs from OpenFOAM
- ROM preparation in OpenFOAM - Steady-state parametric studies
- OpenFOAM is well prepared for algebraic manipulations necessary for ROM construction.

Advantages

- All connected to ROM is done in postprocessing - simulations can be ran in parallel
- All the OpenFOAM capabilities are accessible (including e.g. MRF or turbulence modeling)

Disadvantages

- Extended shapshots have to be stored - a lot of data
- Creation of A_i^ℓ, $i = 1, \ldots, n$ is time consuming

Acknowledgments

This work was supported by the Centre of Excellence for nonlinear dynamic behaviour of advanced materials in engineering CZ.02.1.01/0.0/0.0/15_003/0000493 (Excellent Research Teams) in the framework of Operational Programme Research, Development and Education.

Special thanks to Prof. Michael Hinze from Hamburg university for his inputs and discussions during the preparation of the presented work.

References

[1] Volkwein, S.: Proper Orthogonal Decomposition: Theory and Reduced-Order Modelling. *LN*, University of Konstanz, 2013.

[2] Volkwein, S.: Proper Orthogonal Decomposition: Applications in Optimization and Control

[3] Chaturantabut, S. Sorensen, D. C.: Nonlinear Model Reduction Via Discrete Empirical Interpolation, *SIAM J. Sci. Comput.*, vol. 32, (2010) pp. 2737–2764.

[4] Chaturantabut, S. Sorensen, D. C.: Application of POD and DEIM on Dimension Reduction of Nonlinear Miscible Viscous Fingering in Porous Media, *Math. Comput. Model. Dyn. Syst., (Technical Report: CAAM)*, Rice University, TR09-25

[5] Alla, A. Kutz, J. N.: Nonlinear Model Order Reduction Via Dynamic Mode Decomposition, *preprint*

Thank you for your attention

Algorithm 1 POD basis of rank ℓ with weighted inner product

Require: Snapshots $\{y_j\}_{j=1}^n$, POD rank $\ell \leq d$, symmetric positive-definite matrix of weights $W \in \mathbb{R}^{m\times m}$

1: Set $Y = [y_1, \ldots, y_n] \in \mathbb{R}^{m\times n}$;
2: Determine $\bar{Y} = W^{1/2} Y \in \mathbb{R}^{m\times n}$;
3: Compute SVD, $[\bar{\Psi}, \Sigma, \bar{\Phi}] = \mathrm{svd}(\bar{Y})$;
4: Set $\sigma = \mathrm{diag}(\Sigma)$;
5: Compute $\varepsilon(\ell) = \sum_{i=1}^{\ell} \sigma_i / \sum_{i=1}^{d} \sigma_i$;
6: Truncate $\bar{\Psi} \leftarrow [\bar{\psi}_1, \ldots, \bar{\psi}_l] \in \mathbb{R}^{m\times \ell}$;
7: Compute $\Psi = W^{-1/2} \bar{\Psi} \in \mathbb{R}^{m\times \ell}$;
8: **return** POD basis, Ψ, and ratio $\varepsilon(\ell)$

Notes:

- All the operations on W have to be cheap, including its inversion.
- Do not perform the full SVD, $\Sigma \in \mathbb{R}^{d\times d}$, $d = \mathrm{rank}(\bar{Y})$.

Discrete Empirical Interpolation Method

POD & Greedy algorithm based method for handling non-linearities

Algorithm 2 DEIM

Require: p and matrix $F = [f(t_1, y_1), \dots, f(t_1, y_1)] \in \mathbb{R}^{m \times n}$

1: Compute POD basis $\Phi = [\phi_1, \dots, \phi_p]$ for F
2: $\mathrm{idx} \leftarrow \arg\max_{j=1,\dots,m} |(\phi_1)_{\{j\}}|$;
3: $U = [\phi_1]$ and $\vec{i} = \mathrm{idx}$;
4: **for** $i = 2$ **to** p **do**
5: $\quad u \leftarrow \phi_i$;
6: $\quad$ Solve $U_{\vec{i}} c = u_{\vec{i}}$;
7: $\quad r \leftarrow u - Uc$;
8: $\quad \mathrm{idx} \leftarrow \arg\max_{j=1,\dots,m} |(r)_{\{j\}}|$;
9: $\quad U \leftarrow [U, u]$ and $\vec{i} \leftarrow [\vec{i}, \mathrm{idx}]$;
10: **end for**
11: **return** $\Phi \in \mathbb{R}^{m \times p}$ and index vector, $\vec{i} \in \mathbb{R}^p$

Notes:

- Most of the computational cost is hidden on line 6.

- Let us have rather nice functions defined on a nice domain,

$$\varphi, \tilde{\varphi} \in L^2(\Omega), \quad \Omega \subset \mathbb{R}^3 \ldots \text{bounded, connected, } \ldots$$

- A brief reminder,

$$\langle \varphi, \tilde{\varphi} \rangle_{L^2(\Omega)} = \int_\Omega \varphi \tilde{\varphi} \, \mathrm{d}x, \quad ||\varphi||_{L^2(\Omega)} = \sqrt{\langle \varphi, \varphi \rangle_{L^2(\Omega)}}$$

- Denote Ω^h a FVM discretization of Ω and $\delta\Omega_i^h$ the volume of the i-th cell,

$$\Omega \approx \Omega^h = \bigcup_{i=1}^{\mathtt{nCells}} \Omega_i^h, \quad V(\Omega) \approx V(\Omega^h) = \sum_{i=1}^{\mathtt{nCells}} \delta\Omega_i^h$$

- Introduce a discrete inner product, $\langle \varphi, \tilde{\varphi} \rangle_{L^2_h}$,

$$\langle \varphi, \tilde{\varphi} \rangle_{L^2(\Omega)} = \int_\Omega \varphi \tilde{\varphi} \, \mathrm{d}x \approx \sum_{i=1}^{\mathtt{nCells}} \int_{\Omega_i^h} \varphi \tilde{\varphi} \, \mathrm{d}x = \sum_{i=1}^{\mathtt{nCells}} \varphi_i^h \tilde{\varphi}_i^h \delta\Omega_i^h = \langle \varphi, \tilde{\varphi} \rangle_{L^2_h}$$

- Denote $W = \mathrm{diag}(\delta\Omega_1^h, \ldots, \delta\Omega_{\mathtt{nCells}}^h)$. Hence, $\langle \varphi, \tilde{\varphi} \rangle_{L^2_h} = (\varphi^h)^{\mathrm{T}} W \varphi^h$.