Rektorys’ competition 2014
Simplified model for a rivulet spreading down an inclined wetted plate

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Outline

1. Introduction
   - Why to study rivulet interface
   - Coordinate system
   - Basic principle
   - Assumptions

2. Static rivulet

3. Spreading rivulet

4. Velocity field

5. Conclusions

6. Discussion
Why to study rivulets
Numerous applications in mass transfer and reaction engineering

Hydrodynamics
- Fuel cells
  - water management inside PEMFC fuel cells
- Aerospace engineering
  - in flight formation of rivulets on plane wings

Gas-liquid interface
- Packed columns
  - wetting performance
  - mass transfer coefficients
- Catalytic reactors
  - wetting of the catalyst

[Sulzer ChemTech]

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Used coordinate system
Cartesian coordinate system and basic notations

Notations

- $a \text{[m]}$ ........half-width of the rivulet
- $h \text{[m]}$ .................height
- $l \text{[m]}$ ........ intermediate region length
- $x, y, z \text{[m]}$ ........ coordinate system
- $\alpha [-]$ ........ plate inclination angle
- $\beta [-]$ ........ dynamic contact angle

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Proposed method principle
Parallel between spreading of trickle in time and along the plate

Duffy and Moffat[11]
Description of an uniform rivulet flowing down an inclined plate

Cox-Voinov Law[24]
Description of spreading of the 2D symmetric object in time
Symplifying assumptions
Reduce problem to one spatial and one time coordinate

- Newtonian liquid, $\rho$, $\mu$ and $\gamma$ are constant
- $h_t(t, x, y) = 0$, $Q$ is constant
- $u = (u, v, w)$, $u \gg v \sim w$
- $z = h : u_x = v_y = 0$
- Gravity is the only acting body force.
- The rivulet is shallow. Its dynamic contact angles are assumed small, $\beta(x) \ll 1$, and its GL interface is nearly flat, $h_y(x, y) \ll 1$.
- There is a thin precursor film of height $l$ on the whole studied surface. Thus there is no contact angle hysteresis and $\beta_m = 0$. The height of the precursor film, $l$, can also be taken as the intermediate region length scale well separating the inner and outer solution for the profile shape[24].
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Outline

1. Introduction

2. Static rivulet
   - Gas-liquid interface
   - Liquid flow rate

3. Spreading rivulet

4. Velocity field

5. Conclusions

6. Discussion

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Simplified Navier-Stokes equations
Assumption of very shallow and nearly flat rivulet[11, 12]

Simplified NS

\[0 = -p_x + \rho g \sin \alpha + \mu u_{zz}\]
\[0 = -p_y\]
\[0 = -p_z - \rho g \cos \alpha\]

Used boundary conditions

\[z = 0 : \quad u = u(y, z) = 0\]
\[z = h : \quad p = p_A - \gamma h_{yy} \quad \text{and} \quad u_z = 0\]
\[y = \pm a : \quad h = 0 \quad \text{and} \quad h_y = \pm \tan \beta\]

Notes

\[h = h(y), \quad h_y = \frac{d h}{d y}, \quad p = p(y, z)\]
Legend to results of integration

In the presented case, there exist a physical solution for $\alpha \in (0, \pi)$

**Notation**

The problem decomposes to three cases which will be denoted:

- (i) $\iff \alpha \in (0, \pi/2)$
- (ii) $\iff \alpha = \pi/2$
- (iii) $\iff \alpha \in (\pi/2, \pi)$

**Dimensionless numbers**

$$B = a^2 \frac{\rho g \cos |\alpha|}{\gamma}$$

- $\rho \text{[kg m}^{-3}] \ldots \text{liquid density}$
- $\gamma \text{[N m}^{-1}] \ldots \text{surface tension}$
- $g \text{[m s}^{-2}] \ldots \text{gravitational acceleration}$
Integration results
Rectilinear rivulet gas-liquid interface shape

Gas-liquid interface shape for the three cases

\[ h(\zeta) = \begin{cases} 
\frac{a \tan \beta}{\sqrt{B}} \left( \frac{\cosh \sqrt{B} - \cosh \sqrt{B} \zeta}{\sinh \sqrt{B}} \right) & (i) \\
\frac{a \tan \beta}{2} (1 - \zeta^2) & (ii) \\
\frac{a \tan \beta}{\sqrt{B}} \left( \frac{\cos \sqrt{B} \zeta - \cos \sqrt{B}}{\sin \sqrt{B}} \right) & (iii) 
\end{cases} \]

Scaling of the GL interface height

\[ \tilde{h}(\zeta) = \frac{2h(\zeta)}{a \tan \beta} \]
Effects of Bond number on GL interface shape
Dependence of GL interface shape on the ratio of surface and volumetric forces

Figure: Scheme of the effects of changes in the Bond number on the rivulet GL interface shape. In the case (i), the interface is flattened as the gravity effects grows stronger in comparison with the surface tension. In the case (iii), the increase of $B$ has the narrowing effect on the rivulet GL interface. Case (ii) is depicted for reference. Rivulet contact angle and semi-width are fixed at $\beta = 0.05$ and $a = 0.01$ m.
Liquid volumetric flow rate
Integration of velocity field on a domain of one transversal cut

Velocity field integration

\[
\frac{Q}{a} = \int_{-1}^{1} \int_{0}^{h(\zeta)} u(\zeta, z) \, dz \, d\zeta = \int_{-1}^{1} \int_{0}^{h(\zeta)} \frac{\rho g \sin \alpha}{2\mu} \left(2h(\zeta)z - z^2\right) \, dz \, d\zeta
\]

Dimensionless liquid flow rate

\[
\frac{\mu Q}{a^4 \rho g \sin \alpha \tan^3 \beta} = F(B)
\]

\[
F(B) = \begin{cases} 
\frac{54\sqrt{B} \cosh \sqrt{B} + 6\sqrt{B} \cosh 3\sqrt{B} - 27 \sinh \sqrt{B} - 11 \sinh 3\sqrt{B}}{36B^2 \sinh^3 \sqrt{B}} & (i) \\
\frac{4}{105} & (ii) \\
\frac{27 \sin \sqrt{B} + 11 \sin 3\sqrt{B} - 54\sqrt{B} \cos \sqrt{B} - 6\sqrt{B} \cos 3\sqrt{B}}{36B^2 \sin^3 \sqrt{B}} & (iii)
\end{cases}
\]
Liquid flow rate scaling and properties
Maximal flow rate occurs at $\alpha = \pi/2$

Scaling

$$\tilde{Q} = \frac{105 \mu}{4a^4 \rho g \sin \alpha \tan^3 \beta} Q = \frac{105 \rho g \mu \cos^2 \alpha}{4 \gamma^2 \sin \alpha \tan^3 \beta} \frac{Q}{B^2}$$

Asymptotic behavior

(i) : \( \lim_{B \to \infty} \tilde{Q}(B) = 0 \)

(iii) : \( \lim_{B \to \pi^2} \tilde{Q}(B) = \infty \)
Liquid flow rate scaling and properties
Maximal flow rate occurs at $\alpha = \pi/2$

![Graph (a)](image1)

![Graph (b)](image2)

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Outline

1. Introduction
2. Static rivulet
3. Spreading rivulet
   - Problem definition
   - Basics
   - Gas-liquid interface
4. Velocity field
5. Conclusions
6. Discussion
### Difference from static rivulet

**Changed boundary conditions**

#### Simplified NS

\[
\begin{align*}
0 &= -p_x + \rho g \sin \alpha + \mu u_{zz} \\
0 &= -p_y \\
0 &= -p_z - \rho g \cos \alpha
\end{align*}
\]

#### Used boundary conditions

\[
\begin{align*}
z = 0 & : \quad \mathbf{u}(x, y, z) + \lambda \nabla \mathbf{u} = 0, \quad \lambda \ll 1 \\
z = h & : \quad p = p_A - \gamma \kappa \quad \text{and} \quad u_z = 0 \\
y = \pm a(x) & : \quad h = 0 \quad \text{and} \quad h_y = \mp \tan \beta(x)
\end{align*}
\]

#### Notes

\[
h = h(x, y), \quad h_y = \frac{\partial h}{\partial y}, \quad p = p(x, y, z)
\]

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Cox-Voinov law
Solution of thin film governing equation for spreading of symmetric object[24]

Thin film governing equation - outer and inner

\[ h_t + \frac{\gamma}{3\mu} \frac{\partial}{\partial a} (h^3 h_{aaa}) = 0, \quad h_t + \frac{1}{3\mu} \frac{\partial}{\partial a} \left[ (h^3 + 3\lambda h^2) \gamma h_{aaa} \right] = 0 \]

Cox-Voinov law[24]

\[ \beta(t)^3 = 9 \frac{da(t)}{dt} \frac{\mu}{\gamma} \ln \left( \frac{a(t)}{2e^{2l}} \right) \]

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Basic principle remainder
Combination of Cox-Voinov law with unidirectional rivulet flow description

Spreading of a trickle in time

\[ \beta(t)^3 = 9 \frac{da(t)}{dt} \frac{\mu}{\gamma} \ln \left( \frac{a(t)}{2e^2l} \right) \]

Uniform rivulet interface with prescribed \( Q \)

\[ h(y) = \frac{\tan \beta}{2a} (a^2 - y^2), \quad a \approx \eta \frac{1}{\beta^{3/4}} \]
Transformation from $t$ to $x$

Presence of falling thin liquid film, $l$, on all the plate

From $t$ to $x$ using $u_\tau$

$$u_\tau = \frac{\rho g \sin \alpha}{2\mu} l^2 \implies t = \frac{2\mu}{\rho g l^2 \sin \alpha} x = \omega x$$
Modeling method for the case of no gravity effects
Gas-liquid interface shape defined implicitly by 1 NAE

\[ \beta \ll 1 \implies a \dot{=} \eta \frac{1}{\beta^{3/4}} \]

\[ \beta(t)^3 = 9 \frac{da(t)}{dt} \frac{\mu}{\gamma} \ln \left( \frac{a(t)}{2e^2 l} \right) \sim \beta^{19/4} = -A \frac{d\beta}{dt} \ln \left( \frac{B}{\beta^{3/4}} \right) \]

\[ \beta = \beta(t), \quad A = \frac{27}{4} \frac{\eta \mu}{\gamma}, \quad B = \frac{\eta}{2e^2 l}, \quad \eta = \left( \frac{4\mu Q}{105\rho g \sin \alpha} \right)^{1/4} \]

\[ \beta(0) = \beta_0, \quad t = \omega x \sim x - \frac{\bar{A}}{\beta^{15/4}} \left[ \ln \left( \frac{B}{\beta^{3/4}} \right) - \frac{1}{5} \right] + \bar{C} = 0 \]

\[ \bar{A} = \frac{4}{15} \frac{A}{\omega}, \quad \bar{C} = \frac{4A}{15\omega \beta_0^{15/4}} \left[ \ln \left( \frac{B}{\beta_0^{3/4}} \right) - \frac{1}{5} \right] \]
Modeling method gravity effects

1 ODE with 1 NAE nested in each time step has to be solved numerically

- \( t = \varpi x, \quad \beta(0) = \beta_0 \)

\[
\frac{\mu Q}{a^4 \rho g \sin \alpha \tan^3 \beta} = F(B)
\]

\[
\beta(\varpi x)^3 = \frac{9}{\varpi} \frac{da(\varpi x)}{dx} \frac{\mu}{\gamma} \ln \left( \frac{a(\varpi x)}{2e^2 l} \right)
\]
**B and $\beta$ along the plate**

$\xi = x/L$, $\tilde{\beta} = \beta/\beta_0$, $L = 0.1\text{ m}$, $\beta_0 = 0.05$, $Q = 0.01\text{ ml s}^{-1}$, $\alpha/\pi = \{1/3, 2\pi/3\}$

![Graphs showing B(\xi) and $\tilde{\beta}(\xi)$](image-url)
Outline

1. Introduction
2. Static rivulet
3. Spreading rivulet
4. Velocity field
   - In x axis direction
   - In y and z directions
5. Conclusions
6. Discussion
Velocity in $x$-axis direction
Approximated through the static rivulet description

Dimensional $u(x)$

$$u(x) = (u(x), v(x), w(x))$$

$$u(x) = \frac{\rho g \sin \alpha}{2\mu} \left( 2h(x, y)z - z^2 \right)$$

Scaling

$$\xi = \frac{x}{L}, \quad \zeta = \frac{y}{a(0)}, \quad \tilde{z} = \frac{z}{h(0, 0)}$$

$$\tilde{u} = \frac{u}{u(0)}$$
Contours plot of velocity in $x$-axis direction

Water, $\xi = x/L$, $\tilde{\beta} = \beta/\beta_0$, $L = 0.1$ m, $\beta_0 = 0.05$, $Q = 0.01$ ml s$^{-1}$, $\alpha = \pi/3$

$$\xi = 0.2$$

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Contours plot of velocity in $x$-axis direction

Water, $\xi = x/L$, $\tilde{\beta} = \beta/\beta_0$, $L = 0.1$ m, $\beta_0 = 0.05$, $Q = 0.01$ ml s$^{-1}$, $\alpha = \pi/3$

$$\xi = 0.8$$
Simulation of velocity field in transversal cut
Numerical PIV based technique – connect the dots

For each subrivulet:

- Get the current width of the rivulet, $\zeta_i$, from the Cox-Voinov law.
- Calculate the current maximal rivulet height, $\tilde{h}_i^0$, from description of an uniform rivulet.
- Create a mesh $\Omega_i^h$ on a domain $\Omega_i$,

$$
\Omega_i = \left\{ (\zeta, \tilde{z}) : \zeta \in (0; \zeta_i) \mid \tilde{z} \leq \tilde{h}_i(\zeta) \right\}
$$

$$
\Omega_i^h = \left\{ (\zeta^j, \tilde{z}^k) : \zeta^j \in (0; \zeta_i) \mid \tilde{z}^k \leq \tilde{h}_i(\zeta^j) \right\}_{j=1,\ldots,M_1; k=1,\ldots,M_2}
$$

- Save the current mesh, $\Omega_i^h$.

From the saved meshes, $\Omega_i^h$, $i = 1, \ldots, N$, evaluate the velocity field in the $\zeta - \tilde{h}$ plane (connect the dots).
Simulated velocity field \( \nu \) and \( \omega \) components

Water, \( \xi = x/L, \tilde{\beta} = \beta/\beta_0 \), \( L = 0.1 \text{ m}, \beta_0 = 0.05, Q = 0.01 \text{ ml s}^{-1}, \alpha = \pi/3 \)
Conclusions and Outlook
Description of a spreading rivulet with no need of PDE solving

Practical advantages of proposed algorithm

- When the gravity effects can be neglected, the method is suitable for parallelization
- Usable for qualitative study of dependence of $S_{g-l}$ on process parameters

Accuracy

- Deviation $< 5\%$ in comparison with experimental data [27].

Outlook

- Study of spreading over an edge of the plate (discontinuous change of $\alpha$)
References I

In order of appearance


References II
In order of appearance


References III
In order of appearance


Acknowledgments

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Thank you for your attention