# COMPUTATIONALLY INEXPENSIVE METHOD TO DETERMINE SIZE OF GAS-LIQUID INTERFACIAL AREA OF RIVULET SPREADING ON INCLINED WETTED PLATE 

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## Abstract

Today, the size of the free interface of a liquid objects spreading on substrates can be obtained through various CFD methods. However, all these methods are computationally very demanding, thus not very applicable in the day-to-day engineering practice. In the presented paper, a computationally inexpensive method to determine the size of the gas-liquid interface of the rivulet flowing down an inclined, wetted plate was derived and verified against the experimental data.
Keywords: rivulet, fluid dynamics, liquid spreading.

## 1 Introduction

The size of the gas-liquid interface of the rivulet type flow of a liquid, $S_{g-l}$, is of the key importance throughout many areas of chemical engineering, especially the ones concerning the mass transfer and catalytic reactions [1].

Eventhough $S_{g-l}$ can be obtained from various CFD methods [1,2], such methods are still too complex to be used in the engineering practice and too computationally demanding for the parametric studies of the rivulet free interface behavior.

We will reduce the problem of finding the shape of $(g)-(l)$ interface of the rivulet from solution of the corresponding system of the Navier-Stokes equations to the repeated solution of one nonlinear algebraic equation. This reduction is based on the combination of the Cox-Voinov law [3] and the hydrodynamic description of the rectilinear rivulet published by Duffy and Moffat in [4]. This approach is possible only with the introduction of several simplifying assumptions. The necessary simplifications are listed below.

The used Cartesian coordinate system as well as the most important symbols are presented in Fig. 1.


Figure 1: Used coordinate system with the basics of rivulet spreading notation. $\alpha$ is the plate inclination angle, $\beta$ and $\beta_{m}$ are the apparent and the microscopic contact angles, $a$ is the rivulet half width.

## 2 Simplifying assumptions

The proposed calculation method for the size of the gas-liquid interface of a rivulet flowing down an inclined wetted plate was derived under the following simplifications,

1. The studied liquid is Newtonian, $\rho, \mu$ and $\gamma$ are constant.
2. The rivulet profile shape is constant in time. Furthermore, $Q$ is constant not only in time, but also in all spatial directions.
3. There is no shear between the gas and liquid phases.
4. The liquid velocities in the directions transversal and normal to the plate are negligible in comparison to the one in its longitudinal direction, $u \gg v w$. The inertial effects can be neglected in $y$ and $z$ directions.
5. The gravity is the only acting body force.
6. The rivulet is shallow enough for the gravity effects on its interface shape to be neglected. Also the dynamic contact angles, $\beta=\beta(x)$ are assumed small all along the rivulet.
7. There is a thin precursor film of height $l$ on the whole studied surface. Thus there is no contact angle hysteresis and $\beta_{m}=0$. The height of the precursor film, $l$, can also be taken as the intermediate region length scale well separating the inner and outer solution for the profile shape [3].

## 3 Proposed method

With above listed simplifying assumptions, the parallel between the spreading of a rivulet along the inclined plate and the spreading of a static objects in time can be found.

In the review [3], the Cox-Voinov law for the case of a symmetric 2D object spreading on a horizontal substrate was derived in the form,

$$
\begin{equation*}
\beta(t)^{3}=9 \frac{\mathrm{~d} a(t)}{\mathrm{d} t} \frac{\mu}{\gamma} \ln \left(\frac{a(t)}{2 e^{2} l}\right) \tag{1}
\end{equation*}
$$

with $a(t)$ being the object characteristic dimension. For the case of the narrow, axially symmetric stripe of a liquid, $a(t)$ represents its half width.

The resulting equation is a first order ordinary differential equation for two unknown functions, $\beta(t)$ and $a(t)$.

In the case of a steady rivulet of a liquid flowing and spreading down an inclined plate, the time coordinate in (1) can be transformed in the spatial coordinate, $x$.

Furthermore, for the case of an uniform steady rivulet, with negligible effects of the gravity on the interface shape, Duffy and Moffat have given the solution of the Navier-Stokes equations [4]. Their solution yields the following equations describing the shape the $(g)-(l)$ interface of such a rivulet,

$$
\begin{align*}
h(y) & =\frac{\tan \beta}{2 a}\left(a^{2}-y^{2}\right)  \tag{2}\\
a=\left(\frac{105 \mu Q}{4 \rho g \sin \alpha \tan ^{3} \beta}\right)^{\frac{1}{4}} & \approx \eta \frac{1}{\beta^{3 / 4}}, \quad \eta=\left(\frac{105 \mu Q}{4 \rho g \sin \alpha}\right)^{1 / 4} . \tag{3}
\end{align*}
$$

The importance of the solution of Duffy and Moffat for the problem of finding the $(g)-(l)$ interface shape of a non-uniform rivulet is, that it gives an explicit relation between the rivulet half-width, $a$, and its dynamic contact angle, $\beta$. Substituting $a$ from (3) to (1), one arrives at

$$
\begin{equation*}
\beta^{19 / 4}=-A \frac{\mathrm{~d} \beta}{\mathrm{~d} t} \ln \left(\frac{B}{\beta^{3 / 4}}\right), \quad \beta=\beta(t), \quad A=\frac{27}{4} \frac{\eta \mu}{\gamma}, \quad B=\frac{\eta}{2 e^{2} l} . \tag{4}
\end{equation*}
$$

Solution of (4) yields an implicit relation for $\beta(t)$,

$$
\begin{equation*}
t-\frac{4}{15} \frac{A}{\beta^{15 / 4}}\left[\ln \left(\frac{B}{\beta^{3 / 4}}\right)-\frac{1}{5}\right]+C=0 \tag{5}
\end{equation*}
$$

The integration constant, $C$, is specified by the initial condition, $\beta(0)=\beta_{0}$,

$$
\begin{equation*}
C=\frac{4}{15} \frac{A}{\beta_{0}^{15 / 4}}\left[\ln \left(\frac{B}{\beta_{0}^{3 / 4}}\right)-\frac{1}{5}\right] . \tag{6}
\end{equation*}
$$

Let us take the three phase point of one transversal cut through the rivulet and denote it as $\tau$. The equation (5) describes the movement of $\tau$ in the direction of the $y$ axis in time and the effects of this movement on the shape of the 2D $(g)-(l)$ interface of the chosen transversal cut.

For the description of the rivulet interface shape along the plate, the relation between the movement of $\tau$ in time and the movement of the chosen transversal cut along the plate has to be established.

The presented transformation from time to spatial coordinate arises from the last assumption in Simplifying assumptions. We assume the presence of a precursor film of thickness equal to the intermediate region length scale, $l$, on the whole plate. Neglecting the long-range intermolecular forces, this precursor film can be taken as a free falling film. The point $\tau$ is then considered not to be directly on the three phase line, as there is, in fact none, but in the height $l$ above the plate. Hence $\tau$ is moving along $x$ axis with the speed of

$$
\begin{equation*}
u_{\tau}=\frac{\rho g \sin \alpha}{2 \mu} l^{2} \tag{7}
\end{equation*}
$$

Using this estimate for the speed of $\tau$, the needed transformation is,

$$
\begin{equation*}
t=\varpi x, \quad \varpi=\frac{2 \mu}{\rho g \sin \alpha l^{2}} \tag{8}
\end{equation*}
$$

Substitution for $t$ from (8) to (5) yields the equation for the shape of the rivulet $(g)-(l)$ interface in the dependence on the plate longitudinal coordinate, $x$,

$$
\begin{equation*}
x-\frac{\bar{A}}{\beta^{15 / 4}}\left[\ln \left(\frac{B}{\beta^{3 / 4}}\right)-\frac{1}{5}\right]+\bar{C}=0, \quad \bar{C}=\frac{4}{15} \frac{C}{\varpi} . \tag{9}
\end{equation*}
$$

The proposed method for calculation of the $S_{g-l}$ is based on the approximation of the spreading rivulet by $N$ consecutive subrivulets of the constant width.

## Algorithm:

For each subrivulet, placed at coordinate $i \delta x$ from the plate top do: $\quad i=1,2, \ldots, N, \delta x=L / N$

1. Solve the equation (9) and obtain $\beta_{i}$.
2. Substitute for the $\beta_{i}$ in (3) and get the current rivulet half width, $a_{i}$.
3. Calculate the $S_{g-l}$ for the $i$-th subrivulet using (2) from,

$$
\begin{equation*}
S_{g-l}^{i}=2 \delta x \int_{0}^{a_{i}} h^{i}(y) \mathrm{d} y \tag{10}
\end{equation*}
$$

Calculate the $S_{g-l}$ as sum of the sizes of the $(g)-(l)$ interface of all the subrivulets,

$$
\begin{equation*}
S_{g-l}=\sum_{i=1}^{N} S_{g-l}^{i} \tag{11}
\end{equation*}
$$

## 4 Results and Discussion

For the case of a liquid spreading over a prewetted plate, the proposed method for $S_{g-l}$ calculation provides results within a tolerance of $5 \%$ compared to the experimental data. The comparison between the experimental and theoretical results for such a case is in Fig. 2. The solid curve was obtained by fitting the experimental results using the above described algorithm with $l$ kept as the adjusted parameter. The dot-dashed curve was obtained from the proposed method with $l$ determined on the basis of the experiment.


Figure 2: Comparison of resulting $S_{g-l}$ for fitted $l, l$ determined on the basis of the experiment and experimental data. Measured system is a silicon oil spreading on a wetted, steel plate, $\alpha=60^{\circ}$.

## 5 Conclusion

Even with continuous growth of the computing capacity of the modern computers, there is still need for a simplified solutions to the complex problems of fluid dynamics. Such method for the calculation of the $S_{g-l}$ of a rivulet flowing down an inclined, wetted plate was derived and verified against the experimental data. The presented method can be used for the qualitative study of the dependence of the $S_{g-l}$ on the process parameters. Moreover, as the method provides relatively accurate results, it can be used for the $S_{g-l}$ estimates in the engineering practice.

## Nomenclature

$a \ldots \ldots \ldots \ldots$ half-width of the rivulet, $[\mathrm{m}] \quad x, y, z \ldots \ldots \ldots \ldots \ldots$ coordinate system, $[\mathrm{m}]$
$A, B, C, \bar{C} \ldots \ldots \ldots \ldots \ldots$ constants, $[\mathrm{s},-,-,-]$
$e \ldots \ldots \ldots \ldots \ldots \ldots \ldots$........................ constant, [-]
$g \ldots \ldots . \ldots$. gravitational acceleration, $\left[\mathrm{m} \mathrm{s}^{-2}\right]$
$\qquad$
$l \ldots \ldots$. intermediate region length scale, $[\mathrm{m}]$
L.......................total rivulet length, [m]
N..............number of consecutive cuts, [-]


$t$..............................time coordinate, [s] $u, v, w$. liq. velocity in $x, y$ and $z$ dir., $\left[\mathrm{ms}^{-1}\right]$

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