Math for ChemEng 2015

SODE – Qualitative theory: Solving SODE in form

$$x' = Ax, \quad A \in \mathbb{R}^{n\times n}$$

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November 11 and 13, 2015

Dynamical system

$$\varphi(t, x),\ t \in \mathbb{R},\ x \in \mathcal{S}$$

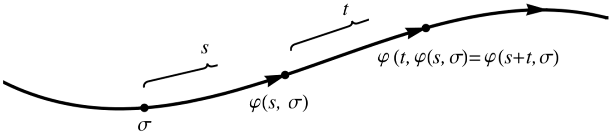
$\varphi : R \times \mathcal{S} \to \mathcal{S}$ .... dynamical system

$\mathcal{S}$ .... state space

Conditions

(i) $\varphi(0, x_0) = x_0, \quad \forall x_0 \in \mathcal{S}$

(ii) $\varphi(t, \varphi(s, x_0)) = \varphi(t + s, x_0), \quad \forall s, t \in \mathbb{R},\ \forall x_0 \in \mathcal{S}$



Flow

Let us have a SODE

$$x'(t) = F(x(t))$$

A mapping

$$\varphi : \mathbb{R} \times \mathcal{S} \to \mathcal{S}$$

for which holds the following,

(i) $\varphi(0, x_0) = x_0, \quad \forall x_0 \in \mathcal{S}$

(ii) $\varphi(t, \varphi(s, x_0)) = \varphi(t + s, x_0), \quad \forall s, t \in \mathbb{R}, \forall x_0 \in \mathcal{S}$

(iii) $\frac{\mathrm{d}\varphi_x(t)}{\mathrm{d}t} = F(\varphi_x(t)), \quad \forall t \in \mathbb{R}, \forall x_0 \in \mathcal{S}$

is called a flow associated to the SODE.

Ex.1: Finding a flow to an ODE

Find a flow to the differential equation $x' = x^p$, $\forall p \in \mathbb{N}$

Solution to Ex.1

(Solve the ODE and specify C in the manner that (i)–(iii) hold)

Case: $p = 1$

Find a general solution

$$x' = x \to x(t, C) = C\mathrm{e}^t$$

Specify C in a way that (i)–(iii) holds

$$\begin{array}{rcc} \text{(i)} & \to & x(0, C) = C\mathrm{e}^0 = C = x_0 \\ \text{(ii)} & \to & x(t, x(s, x_0)) = x_0\mathrm{e}^s\mathrm{e}^t = x_0\mathrm{e}^{t+s} = x(t+s, x_0) \\ \text{(iii)} & \to & \frac{\mathrm{d}x(t,x_0)}{\mathrm{d}t} = x_0\mathrm{e}^t = x(t, x_0) = F(x(t, x_0)) \end{array}$$

Case: $p \geq 2$

Try it as a homework. It may happen, that the flow cannot be found.

Ex.2: Competitive reactions

Let us have a reaction scheme,

$$\begin{aligned} a &\xrightarrow{k_1} b \xrightarrow{k_2} c \\ a &\xrightarrow{k_3} e \end{aligned}$$

generating the corresponding differential equations,

$$x' = \begin{pmatrix} a' \\ b' \\ c' \\ e' \end{pmatrix} = \begin{pmatrix} -k_1 a - k_3 a \\ k_1 a - k_2 b \\ k_2 b \\ k_3 a \end{pmatrix} = \begin{pmatrix} -k_1 - k_3 & 0 & 0 & 0 \\ k_1 & -k_2 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ k_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix} = Ax$$

Find dependence of contcentrations on time corresponding to the following parameter values,

$$a_0 = 1,\ b_0 = c_0 = e_0 = 0,\ k_1 = 1, k_2 = 1/2, k_3 = 1/10$$

Solution algorithm

(eigenvalues $\rightarrow$ eigenvectors $\rightarrow$ general solution $\rightarrow$ particular solution)

$$x' = Ax, \quad x \in \mathbb{R}^n,\, A \in \mathbb{R}^{n \times n}$$

Eigenvalues, λ_i, $i = 1, 2, \ldots, n$

Eigenvectors, h_i, $i = 1, 2$

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$$x' = Ax, \quad x \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n}$$

Eigenvalues, $\lambda_i,\ i = 1, 2, \ldots, n$

1. Construct characteristic polynomial and find its roots,
   $\det\left(A - \lambda E\right) = 0 \rightarrow \lambda_1,\ \lambda_2, \ldots, \lambda_n$

Eigenvectors, $h_i,\ i = 1, 2$

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2. $n = 2 \to 3$ possibilities

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   (i) $\lambda_{1,2} \in \mathbb{R}, \quad \lambda_1 \neq \lambda_2$

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What changes when $n \geq 4$?

Eigenvectors, $h_i,\ i = 1, 2$

Solution algorithm

(eigenvalues $\rightarrow$ eigenvectors $\rightarrow$ general solution $\rightarrow$ particular solution)

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Eigenvectors, $h_i,\ i = 1, 2$

case (i) We known this, $\lambda_1 \rightarrow h_1,\ \lambda_2 \rightarrow h_2,\ h_1, h_2 \in \mathbb{R}^2$

Solution algorithm

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case (ii) We known this, $\lambda_1 \rightarrow h_1 = u + iv,\ h_2 = u - iv,\ u, v \in \mathbb{R}^2$

Solution algorithm

(eigenvalues → eigenvectors → general solution → particular solution)

$$x' = Ax, \quad x \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n}$$

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What changes when $n \geq 4$?

Eigenvectors, $h_i,\ i = 1, 2$

case (i) We known this, $\lambda_1 \to h_1,\ \lambda_2 \to h_2,\ h_1, h_2 \in \mathbb{R}^2$

case (ii) We known this, $\lambda_1 \to h_1 = u + iv,\ h_2 = u - iv,\ u, v \in \mathbb{R}^2$

case (iii) Hic sunt leones

Solution algorithm

eigenvectors – case of $\lambda_1 = \lambda_2 = \lambda_0 \to$ general solution

Possibility 1: $\dim \mathcal{N}(A - \lambda_0 E) = 2$

Same as case (i), $\lambda_0 \to h_1$, $\lambda_0 \to h_2$, $h_1, h_2 \in \mathbb{R}^2$

Possibility 2: $\dim \mathcal{N}(A - \lambda_0 E) = 1$

There are no 2 linearly independent eigenvectors corresponding to λ_0

$\Longrightarrow$ **generalized eigenvector**, k

1. $(A - \lambda_0 E)h = 0 \to h$
2. $(A - \lambda_0 E)k = h \to k$

General solution

case (i) $x_G(t, C) = C_1 e^{\lambda_1 t} h_1 + C_2 e^{\lambda_2 t} h_1$

case (ii) $x_G(t, C) = C_1 e^{at}(u \cos bt - v \sin bt) + C_2 e^{at}(v \cos bt + u \sin bt)$

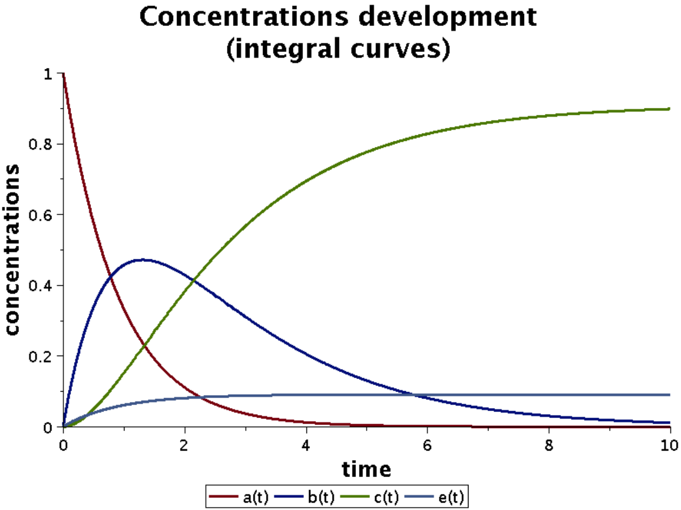
case (iii) $x_G(t, C) = C_1 e^{\lambda_0 t} h + C_2 e^{\lambda_0 t}(th + k)$

Solution to Ex.2

(eigenvalues $\rightarrow$ eigenvectors $\rightarrow$ general solution $\rightarrow$ particular solution)

Solution to Ex.2

(eigenvalues → eigenvectors → general solution → particular solution)



Note: Might be too big of a leap. Lets start with something smaller and return to this later.

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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Eigenvalues

1. ch. p.: $\lambda^2 - \operatorname{Tr} A\, \lambda + \det A = 0$

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Eigenvalues

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Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr} A\, \lambda + \det A = 0 \quad \rightarrow \lambda^2 - 6\lambda + 5 = 0$
2. $\lambda_1 = 5,\ \lambda_2 = 1$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36 - 20}}{2} = \frac{6 \pm 4}{2} \begin{matrix} \nearrow 5 \\ \searrow 1 \end{matrix}$$

Assignement

Solve the Cauchy problem,

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Eigenvectors

$\lambda_1 = 5, \lambda_2 = 1$

1. $(A - \lambda_1 E)h_1 = 0$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 5, \lambda_2 = 1$

1. $(A - \lambda_1 E)h_1 = 0$

$$\left[\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h_1 = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} h_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies h_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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2. $(A - \lambda_2 E)h_2 = 0$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 5, \lambda_2 = 1$

1. $(A - \lambda_1 E)h_1 = 0$

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2. $(A - \lambda_2 E)h_2 = 0$

$$\left[\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h_2 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} h_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies h_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 5, \lambda_2 = 1$

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General solution

Ex.3

case (i), $\lambda_{1,2} \in \mathbb{R}, \quad \lambda_1 \neq \lambda_2$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = \sum_{i=1}^{n(=2)} C_i \mathrm{e}^{\lambda_i t} h_i$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = \sum_{i=1}^{n(=2)} C_i e^{\lambda_i t} h_i$$

1. $\lambda_1 = 5,\ \lambda_2 = 1$

$$\lambda_1 = 5,\ h_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \lambda_2 = 1,\ h_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

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Solve the Cauchy problem,

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2. General solution

$$x_G(t, C) = C_1 \mathrm{e}^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \mathrm{e}^{t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 \mathrm{e}^{5t} - 3C_2 \mathrm{e}^{t} \\ C_1 \mathrm{e}^{5t} + C_2 \mathrm{e}^{t} \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

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Particular solution

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1. Substitute from the initial condition

$$x_G(0, C) = C_1 \cdot 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cdot 1 \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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2. Construct and solve set of LAE

$$\begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{GE}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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2. Construct and solve set of LAE

$$\begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{GE}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

3. Particular solution: $x_P(t) = \frac{1}{4}(\mathrm{e}^{5t} + 3\mathrm{e}^{t}, \mathrm{e}^{5t} - \mathrm{e}^{t})^{\mathrm{T}}$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.:

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr} A\, \lambda + \det A = 0$

Ex.4

case (ii), $\lambda_{1,2} \in \mathbb{C}, \quad \lambda_{1,2} = a \pm ib,\ a, b \in \mathbb{R}$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr} A\, \lambda + \det A = 0 \ \ \rightarrow \lambda^2 - 4\lambda + 5 = 0$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr} A\, \lambda + \det A = 0 \quad \rightarrow \lambda^2 - 4\lambda + 5 = 0$
2. $\lambda_1 = 2 + i$, $\lambda_2 = 2 - i$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr}\, A\, \lambda + \det A = 0 \;\; \rightarrow \lambda^2 - 4\lambda + 5 = 0$
2. $\lambda_1 = 2 + i$, $\lambda_2 = 2 - i$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2}$$

Eigenvectors

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 2 + i$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 2 + i$

1. $(A - \lambda_1 E)h_1 = 0$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 2 + i$

1. $(A - \lambda_1 E)h_1 = 0$

$$\left[\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} - (2+i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h_1 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} h_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 2 + i$

1. $(A - \lambda_1 E)h_1 = 0$

$$\left[\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} - (2+i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h_1 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} h_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. h_1

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 2 + i$

1. $(A - \lambda_1 E)h_1 = 0$

$$\left[\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} - (2+i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h_1 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} h_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. h_1

$$h_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u + iv$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_1 = 2 + i$

1. $(A - \lambda_1 E)h_1 = 0$

$$\left[\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} - (2+i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h_1 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} h_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. h_1

$$h_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = u + iv$$

General solution

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 \mathrm{e}^{at}(u \cos bt - v \sin bt) + C_2 \mathrm{e}^{at}(v \cos bt + u \sin bt)$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 \mathrm{e}^{at}(u \cos bt - v \sin bt) + C_2 \mathrm{e}^{at}(v \cos bt + u \sin bt)$$

1. $\lambda_1 = 2 + i \to a = \Re(\lambda_1) = 2,\ b = \Im(\lambda_1) = 1$

Ex.4

case (ii), $\lambda_{1,2} \in \mathbb{C}, \quad \lambda_{1,2} = a \pm ib,\ a, b \in \mathbb{R}$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 \mathrm{e}^{at}(u \cos bt - v \sin bt) + C_2 \mathrm{e}^{at}(v \cos bt + u \sin bt)$$

1. $\lambda_1 = 2 + i \rightarrow a = \Re(\lambda_1) = 2,\ b = \Im(\lambda_1) = 1$
2. $$h_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow u = \Re(h_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} v = \Im(h_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 e^{at}(u \cos bt - v \sin bt) + C_2 e^{at}(v \cos bt + u \sin bt)$$

1. $\lambda_1 = 2 + i \rightarrow a = \Re(\lambda_1) = 2,\ b = \Im(\lambda_1) = 1$
2. $$h_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow u = \Re(h_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} v = \Im(h_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
3. General solution

$$x_G(t, C) = C_1 e^{2t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right] + C_2 e^{2t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right]$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 e^{at}(u \cos bt - v \sin bt) + C_2 e^{at}(v \cos bt + u \sin bt)$$

1. $\lambda_1 = 2 + i \to a = \Re(\lambda_1) = 2,\ b = \Im(\lambda_1) = 1$
2. $$h_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to u = \Re(h_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ v = \Im(h_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
3. General solution

$$x_G(t, C) = C_1 e^{2t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t \right] + C_2 e^{2t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right]$$

Particular solution

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Particular solution

$$x_G(t, C) = C_1 \mathrm{e}^{at}(u \cos bt - v \sin bt) + C_2 \mathrm{e}^{at}(v \cos bt + u \sin bt)$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Particular solution

$$x_G(t, C) = C_1 e^{at}(u \cos bt - v \sin bt) + C_2 e^{at}(v \cos bt + u \sin bt)$$

1. Substitute from the initial condition

$$x_G(0, C) = C_1 \cdot 1 \cdot \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot 1 \right] + C_2 \cdot 1 \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 1 \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Particular solution

$$x_G(t, C) = C_1 \mathrm{e}^{at}(u \cos bt - v \sin bt) + C_2 \mathrm{e}^{at}(v \cos bt + u \sin bt)$$

1. Substitute from the initial condition

$$x_G(0, C) = C_1 \cdot 1 \cdot \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot 1 \right] + C_2 \cdot 1 \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 1 \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. Construct and solve set of LAE

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{GE}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Particular solution

$x_G(t, C) = C_1 \mathrm{e}^{at}(u \cos bt - v \sin bt) + C_2 \mathrm{e}^{at}(v \cos bt + u \sin bt)$

1. Substitute from the initial condition

$$x_G(0, C) = C_1 \cdot 1 \cdot \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot 1 \right] + C_2 \cdot 1 \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 1 \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. Construct and solve set of LAE

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{GE}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

3. Particular solution: $x_P(t) = \mathrm{e}^{2t}(\cos t, \sin t)^{\mathrm{T}}$

Ex.5

case (iii), $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 = \lambda_2 = \lambda_0$, **Note:** generalized eigenvector

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.:

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr} A \, \lambda + \det A = 0$

Ex.5

case (iii), $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 = \lambda_2 = \lambda_0$, **Note:** generalized eigenvector

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr} A\, \lambda + \det A = 0 \quad \rightarrow \lambda^2 - 8\lambda + 16 = 0$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr} A\, \lambda + \det A = 0 \quad \rightarrow \lambda^2 - 8\lambda + 16 = 0$
2. $\lambda_1 = \lambda_2 = \lambda_0 = 4$

$$\lambda_{1,2} = \frac{8 \pm \sqrt{64 - 64}}{2} = 4$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues

1. ch. p.: $\lambda^2 - \mathrm{Tr} A\, \lambda + \det A = 0 \quad \rightarrow \lambda^2 - 8\lambda + 16 = 0$
2. $\lambda_1 = \lambda_2 = \lambda_0 = 4$

$$\lambda_{1,2} = \frac{8 \pm \sqrt{64 - 64}}{2} = 4$$

Eigenvectors

Ex.5

case (iii), $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 = \lambda_2 = \lambda_0$, **Note:** generalized eigenvector

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_0 = 4$

Ex.5

case (iii), $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 = \lambda_2 = \lambda_0$, **Note:** generalized eigenvector

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_0 = 4$

1. $(A - \lambda_0 E)h = 0$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_0 = 4$

1. $(A - \lambda_0 E)h = 0$

$$\left[\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} h = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_0 = 4$

1. $(A - \lambda_0 E)h = 0$

$$\left[\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} h = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2. $(A - \lambda_0 E)k = h$ – generalized eigenvector

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_0 = 4$

1. $(A - \lambda_0 E)h = 0$

$$\left[\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} h = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2. $(A - \lambda_0 E)k = h$ – generalized eigenvector

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} k = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies k = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvectors

$\lambda_0 = 4$

1. $(A - \lambda_0 E)h = 0$

$$\left[\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] h = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} h = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2. $(A - \lambda_0 E)k = h$ – generalized eigenvector

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} k = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies k = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

General solution

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 \mathrm{e}^{\lambda_0 t} h + C_2 \mathrm{e}^{\lambda_0 t}(ht + k)$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 \mathrm{e}^{\lambda_0 t} h + C_2 \mathrm{e}^{\lambda_0 t}(ht + k)$$

1. $\lambda_1 = \lambda_2 = \lambda_0 = 4$

$$\lambda_0 = 4,\ h = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ k = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 \mathrm{e}^{\lambda_0 t} h + C_2 \mathrm{e}^{\lambda_0 t}(ht + k)$$

1. $\lambda_1 = \lambda_2 = \lambda_0 = 4$

$$\lambda_0 = 4,\ h = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ k = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

2. General solution

$$x_G(t, C) = C_1 \mathrm{e}^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \mathrm{e}^{4t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] = \mathrm{e}^{4t} \begin{pmatrix} C_1 + C_2(t-1) \\ C_1 + C_2 t \end{pmatrix}$$

Ex.5

case (iii), $\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 = \lambda_2 = \lambda_0$, **Note:** generalized eigenvector

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

General solution

$$x_G(t, C) = C_1 \mathrm{e}^{\lambda_0 t} h + C_2 \mathrm{e}^{\lambda_0 t}(ht + k)$$

1. $\lambda_1 = \lambda_2 = \lambda_0 = 4$

$$\lambda_0 = 4,\ h = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ k = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

2. General solution

$$x_G(t, C) = C_1 \mathrm{e}^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \mathrm{e}^{4t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] = \mathrm{e}^{4t} \begin{pmatrix} C_1 + C_2(t-1) \\ C_1 + C_2 t \end{pmatrix}$$

Particular solution

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Particular solution

$$x_G(t, C) = C_1 \mathrm{e}^{\lambda_0 t} h + C_2 \mathrm{e}^{\lambda_0 t}(ht + k)$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Particular solution

$$x_G(t, C) = C_1 \mathrm{e}^{\lambda_0 t} h + C_2 \mathrm{e}^{\lambda_0 t}(ht + k)$$

1. Substitute from the initial condition

$$x_G(0, C) = C_1 \cdot 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cdot 1 \cdot \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 0 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Particular solution

$$x_G(t, C) = C_1 e^{\lambda_0 t} h + C_2 e^{\lambda_0 t}(ht + k)$$

1. Substitute from the initial condition

$$x_G(0, C) = C_1 \cdot 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cdot 1 \cdot \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 0 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. Construct and solve set of LAE

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{GE}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Assignement

Solve the Cauchy problem,

$$x' = Ax, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}, \quad x(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Particular solution

$$x_G(t, C) = C_1 e^{\lambda_0 t} h + C_2 e^{\lambda_0 t}(ht + k)$$

1. Substitute from the initial condition

$$x_G(0, C) = C_1 \cdot 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cdot 1 \cdot \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 0 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. Construct and solve set of LAE

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{GE}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

3. Particular solution: $x_P(t) = e^{4t}(1 - t, -t)^{\mathrm{T}}$

Thank you for your attention